Journal of Innovation in Psychology, Education and Didactics
ISSN: 2247-4579, E-ISSN: 2392-7127
http://www.jiped.ub.ro/
Covered in : EBSCO, CEEOL, ProQuest, DOAJ, Scipio,
International Innovative Journal Impact Factor, CiteFactor,
EuroPub database, Open Academic Journals Index, ResearchBib,
Universal Impact Factor
2022, Volume 26, Issue 2, pages: 187-196, doi:10.29081/JIPED.2022.26.2.05

Contributions of Mathematical Modelling for Teaching Systems of Equations

Elena Roxana ARDELEANU [1]*, Otilia LUNGU [2]

Received: 20 September 2022/ Accepted: 23 October 2022/ Published: 26 October 2022

Abstract

The main aim of the study was to identify the ability to use mathematical notions in everyday life and the capacity to make connections starting from the poor results that Romania registers in the PISA tests. On the other hand, the results of the research applied to a group of 200 first-year students confirmed the inability to make connections between theory and real life. Linear algebra is an important discipline and very difficult for most students. The main obstacle is modeling the real problems in abstract concepts of algebra. So it is very important to help students understand how to "translate" real-life problems in linear algebra concepts and then how to solve them using different examples and finally presenting the global theory. This research shows how it is possible for a professor to teach the algorithm for problem-solving in linear algebra and how to guide the students to find the solutions. We present the possibility of introducing the concepts of the compatibility of the systems of linear equations using examples and graphics.

Key words: Geometrical representation; mathematical modelling, systems of equations,

How to cite: Ardeleanu, R., Lungu, O. (2022). Contributions of Mathematical Modelling for Teaching Systems of Equations. *Journal of Innovation in Psychology, Education and Didactics*. *Journal of Innovation in Psychology, Education and Didactics*, *26*(2), 187-196. doi:10.29081/JIPED.2022.26.2.05.

[1] Lecturer PhD, Vasile Alecsandri University of Bacau, Romania, E-mail: rardeleanu@ub.ro
[2] Lecturer PhD, Vasile Alecsandri University of Bacau, Romania, E-mail: otilia.lungu@ub.ro
* Corresponding author

1. Introduction

According to the website of the Romanian Ministry of Education (www.edu.ro), the Programme for International Student Assessment (PISA) has been taking place in Romania since 2000. Within the program, the most extensive international evaluation in the field of education is carried out, an evaluation in which more than 70 countries participate. The purpose of the tests is to measure the skills needed by students at the end of the compulsory education cycle (15-16 years) in the fields of reading, sciences, and mathematics from the perspective of life preparation and integration into the labor market.

Regarding the results obtained in Romania in mathematics, we note that in 2018 they are higher compared to those in 2006 and 2009 but decreasing compared to those obtained in 2012 and 2015. Table 1 shows the scores recorded by Romania at the PISA Mathematics Tests from 2000 to 2018 and the averages recorded by the participating countries.

Table 1. The scores recorded by Romania and the average scores at PISA Mathematics Tests from 2000 to 2018

Year	Score Romania	The average score in math	Position
2000	426	500	-
2003	absent	-	-
2006	415	498	42
2009	427	496	42
2012	445	494	45 of 65
2015	444	490	46 of 70
2018	430	489	52 of 78

In addition, Romania's score is below the average score calculated by the Organization for Economic Cooperation and Development (OECD). According to www.edu.ro, the main factor influencing student performance in the PISA tests is the socioeconomic status of the student's families. The statistical data recorded also showed significant disparities in results between schools. What does this mean? That there is a tendency to group students with very good results in performing schools. In addition, it was observed that the correlation between the level of preparation of the students and the level of education of the teachers is a weak one, although more than 49% of the teachers in the participating schools (the average of the participating countries being of 44%) have taken master's courses. The teacher's attitude towards the act of teaching and the enthusiasm shown in the didactic activity, however, significantly influence the students' results.

The question remains: why Romania registers poor results in the Pisa mathematics tests? The answer is as simple as possible: the Pisa tests analyze the degree of understanding and the ability to use mathematical notions in everyday life, the ability to reason quantitatively to understand relationships or interdependencies in the form of graphs, charts, or tables. All the while, blasé teachers without enthusiasm for the act of teaching emphasize solving algorithms, and formulas, without the intention of connecting with real life and answering questions like "Why do we need this?".

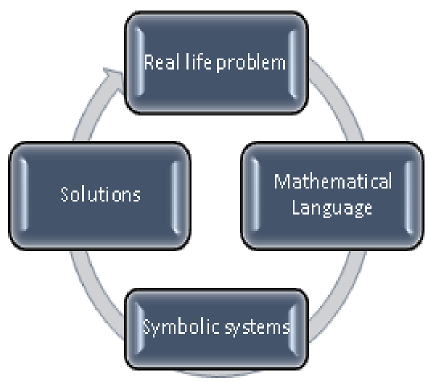
Understanding Science in general and mathematics, in particular, is necessary not only for those whose careers directly depend on it but also for those who wish to make assumed decisions in their personal life. Math problems in school can help us solve personal problems such as calculating a balanced diet, calculating the risk of certain investments, finding the optimum in a personal car travel problem, and so on. Unlike other exact sciences such as biology, physics, and chemistry, the connections of mathematics with reality are not so easy to notice, although the academician Solomon Marcus called mathematics “a bridge between all disciplines” (Marcus, 2010). Emphasizing the strong interdisciplinary character of mathematics creates strong motivations for learning it. The principle of correlation between theory and practice emphasizes the fact that all acquired notions will be capitalized.

We also analyzed the results obtained by a group of 200 students in Engineering in the first course, in the first year, during an initial test regarding systems of linear equations. The main aim of the study was to identify the ability to use mathematical notions in everyday life and the capacity to make connections. The method of data collection is a mathematical test and a questionnaire, which allows for obtaining quantitative data and their analysis using statistical information programs.

2. Systems of Equations: Different Views

2.1. About mathematical modeling

According to Haines and Crouch (Haines & Crouch, 2007), mathematical modeling is, on the one hand, a cyclical process represented in the following diagram.



On the other hand, Vesschaffel, Greer, and De Corte (2002) see mathematical modeling as a process in which day-by-day problems are reformulated by using mathematics. The ability to translate a real-life situation into mathematical language implies that students have other soft skills such as estimation, spatial reasoning, and interpretation beyond hard skills such as computational (Lehrer & Schauble, 2003). So, starting from a concrete situation, students must go through a process named modeling to obtain an abstract model, which is the product of modeling (Sriraman, 2006).

In 2016, Biembengut (2016) specify that "Modelling is a teaching method with research in school boundaries and spaces, in any subject and stage of schooling: from the initial years of elementary school to the end of higher education or postgraduate courses." When the topic of systems is brought up in upper elementary school, students face real modeling concepts. Unfortunately, this cycle is interrupted since high school, and after, in the higher education cycle, abstraction and theorizing are used a lot.

Starting from the mathematical modeling cycle proposed by Blum and Leiβ in 2007 (Blum & Leiβ, 2007) we identify seven steps: to present the real situation, to structure this real situation, to get a real model, to translate it into a mathematical model, to find mathematical results, to compare and explain real results.

Equations and systems of equations are closely related to their application in everyday life. We will approach the topic of solving systems of equations from two perspectives: what need students to know about systems of equations and their solutions to be able to use them in real life and the second perspective, how students deal with systems in an upper algebra course.

2.2. Teaching systems in secondary school

Next, we present to our students a real situation. Every year the municipality organizes a sports marathon for children and adults in our city. The funds collected that year are donated to a school in the city for the rehabilitation of the sports hall. Knowing how many participants there were last year and what amount was collected, the organizers want to know whom they should address in the campaign to promote the event: children or adults. Students will determine how many children participated and how many adults to find out who the promotion should be targeted.

This last year approximately 200 people participated in the event and collected 5200 RON. The price of a ticket for a child is 20 RON and the price of a ticket for an adult is 30 RON. Establish to whom this year's promotional campaign should be addressed.

To make the right decision, the number of participating children and the number of adults must be determined. So, from a mathematical point of view, we have two unknowns that must satisfy the two relationships indicated: one related to the number of participants and one related to the amount received. If we denote by x the number of children and by y the number of adults we deduce the next two equations system: $\begin{cases} x+y=200 \\ 20x+30y=5200 \end{cases}$. Using the substitution method we express x as the total number of people minus the number of adults: $x=200-y$. Replacing in the second equation we get $20(200-y)+30y=5200$. We solve this equation and find $y=120$. Then it is obvious that $x=80$. Given that the number of adults is greater than that of children, the campaign should be addressed especially to adults, either because they want to exercise, or because they want to train their child.

Geometrically, the 2 equations of the system represent the equations of 2 straight lines. Their point of intersection indicates the solution of the system which is uniquely determined (see Figure 1).

To complicate and analyze the problem we have several options. For example, we can try to make a forecast. If this year the number of participants will reach 250 persons, without increasing the price of tickets, how many children and how many adults should participate to have the same amount collected? In this case, only the first equations will change. We can ask students to give the solution using the geometric representation and after to solve the system.
Or, we can slightly modify the equations to analyze other types of solutions. So, we ask the students to analyze the solution of the system $\begin{cases} x+y=200 \\ 20x+20y=6000 \end{cases}$. That means we suppose that we have the same number of participants, the price for one ticket is 20 RON for children and adults, and the revenue will be 6000 RON. From a geometric point of view, the equations will be represented as two parallel lines (*Figure 2*). In this case, it is deduced that the problem has no solutions. Afterward, it can also be checked by calculation, arriving at a relationship of the form $200=300$ which is impossible.

And in the third option, we consider the case where the number of participants is the same, the ticket price is still 20 RON for both children and adults, but we estimate an income of 4000 RON. In this situation, the students will notice that geometrically representing the two equations they will get two lines confused. like this but the system does not admit a unique solution. In this case, the system accepts multiple solutions (see Figure 3).

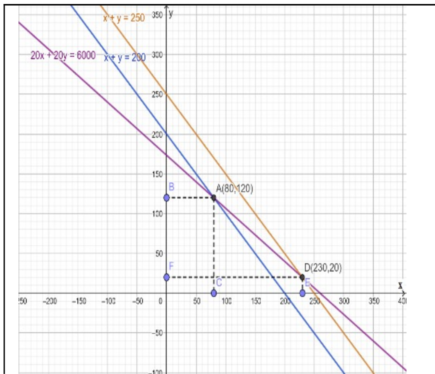


Fig.1 System with a unique solution

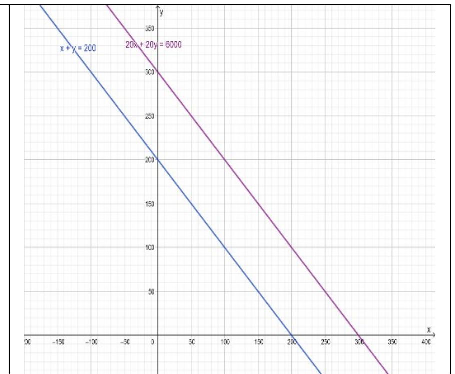


Fig.2 System without a solution

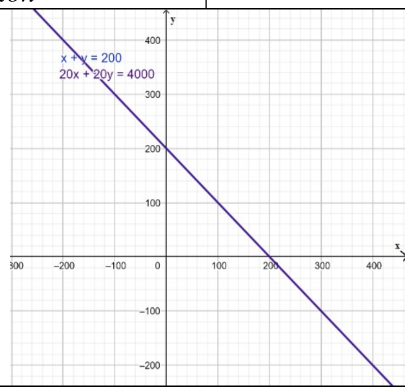


Fig.3 System with multiple solutions

Analyzing these situations, students will understand the fact that not all systems of equations admit a single solution, that there are situations when they have multiple solutions or situations when they do not admit a solution. Thus, students carry out a system compatibility study.

2.3. Teaching systems in high school

Later they analyze the compatibility of a linear system of equations without solving the problem. In high school, the number of unknowns and equations in a system will increase. In this way, we will study systems of m equations with n unknowns. We support the avoidance of expositive methods in teaching linear systems of equations. As the rigorous study of such types of systems is usually done in the 11th grade, teachers consider their students mature enough for lectures. Therefore, according to the curriculum and the textbooks, the lesson is presented following some steps.

The teacher writes the general form of a system of n equations and n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\quad\quad \dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}.$$

Next, the Cramer system is defined and it is given Cramer's rule to determine the solution of such a system. In the next lesson, a lecture on the study of the compatibility of systems of m equations with n unknowns should capture the students' attention. Consequently, the Kronecker-Capelli theorem and Rouche's theorem are stated. A solution algorithm is then compiled and its operation is exemplified. At the end of the chapter, although the teacher has taught scientifically correct lessons, the student does not know how to apply solving algorithms and does not understand their usefulness. The reason lies in choosing an incorrect teaching method. The lecture or exposition used in this case represents a traditional teaching method in which the student is viewed as an object. Students listen to the teacher, and write everything down in their notebooks but do not ask questions, and do not actively participate in class. They do not understand the value and usefulness of the concepts taught. We believe that the use of interactive teaching methods such as discovery learning, and mathematical modeling would be much more efficient in this case. Besides the theoretical notions, they could make connections with problems from everyday life.

High school students should be able to use the elements of algebra to describe and solve concrete life situations. Based on these observations, we propose another way of teaching the chapter under discussion. In the beginning, a real situation is formulated. As part of a volunteer activity, high school students give supplies to children from village schools: pens, notebooks, and colored pencils. All packages will be identical. Each package will contain 30 supplies and will cost 160 RON. Knowing that a pen costs 7 RON, a notebook costs 8 RON and a colored pencil costs 3 RON, and that it needs 2 times more notebooks than pens, determine how many supplies of each kind a package will contain. Starting from this real situation, the mathematical model must be formulated. In this sense, we will denote with *x, y*, and *z* the unknowns that appear in the system: the number of pens, the number of notebooks, respectively the number of colored pencils. Thus, the presented situation is reformulated mathematically as follows: solve the following system of equations:

$$\begin{cases} 7x + 8y + 3z = 160 \\ x + y + z = 30 \\ 2x - y = 0 \end{cases}.$$

Using the substitution method already known, students substitute y from the last equation into the first two equations and solve a system of two equations with two unknowns. Students determine the triplet solution $x = 5, y = 10, z = 15$. In conclusion, each package will contain 5 pens, 10 notebooks and 15 colored pencils.

In order to analyze other forms of the system, the teacher can subject to attention a slightly modified situation. The problem is reformulated. It is considered that a pen costs 7 RON, a notebook 4 RON, and a pencil 5 RON. The new system will be

$$\begin{cases} 7x + 4y + 5z = 160 \\ x + y + z = 30 \\ 2x - y = 0 \end{cases}.$$

Using the substitution method, a system of two equations with two unknowns will be obtained:

$$\begin{cases} 15x + 5z = 160 \\ 15x + 5z = 150 \end{cases}.$$

It is noted that the system is incompatible because $160 \neq 150$. Under these conditions, such packages with supplies cannot be purchased.

One step further. If the amount for the purchase of a package is changed from 160 RON to 150 RON, then the new system is obtained: $\begin{cases} 15x + 5z = 150 \\ 15x + 5z = 150 \end{cases}$. Students must notice that the system reduces to a single equation equivalent to $3x + z = 30$. This equation with 2 unknowns admits an infinity of solutions in the set of real numbers and 11 solutions in the set of natural numbers:

$$\begin{array}{lll} (z = 0, x = 10) & (z = 3, x = 9) & (z = 6, x = 8) \\ (z = 9, x = 7) & (z = 12, x = 6) & (z = 15, x = 5) \\ (z = 18, x = 4) & (z = 21, x = 3) & (z = 24, x = 2) \\ (z = 27, x = 1) & (z = 30, x = 0). & \end{array}$$

Taking into account the real situation described and the fact that all packages must contain the three types of supplies, only 9 solutions remain accepted in the set of natural numbers. Analyzing the three situations presented, the student pays attention to the study of the system, to the importance of analyzing its compatibility, and the teacher can begin to define the required notions and state the theorems necessary for studying systems of linear equations. Although the three systems have almost the same shape, we notice that one of them does not admit any solution and another admits several solutions. Therefore, a study of the compatibility of the systems is required. Now the statement of the Kronecker Capelli and Rouche theorems acquires a different meaning for students.

We used this way of presentation and in the next step we described the general theory of systems with different methods and algorithms for studying the compatibility and for getting the solution (direct methods- elimination, substitution-, Cramer's Rule, Gauss eliminations method, Rouche method). After this experiment, we asked again our group to solve some problems. This time 173 gave correct solutions.

3. Application

During this research the students had to analyze the next situation:

"In a pizza shop, they want to prepare a new type of pizza with 3 types of cheese: mozzarella, parmesan, and gouda. The price of the cheeses used for 1 kilo of pizza is 63 RON. The cost per kilo of these cheeses is 70 RON, 60 RON, and 50 RON, respectively. The amount of mozzarella cheese is to be twice the amount of Gouda cheese."

They had to answer the following requirements:

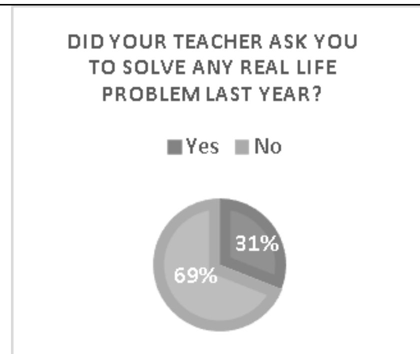
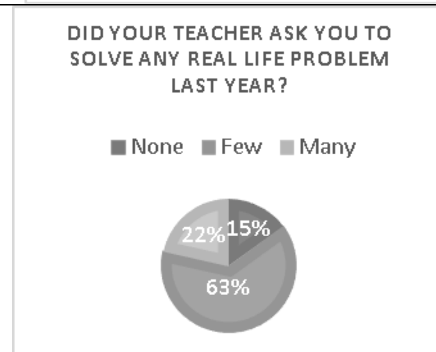
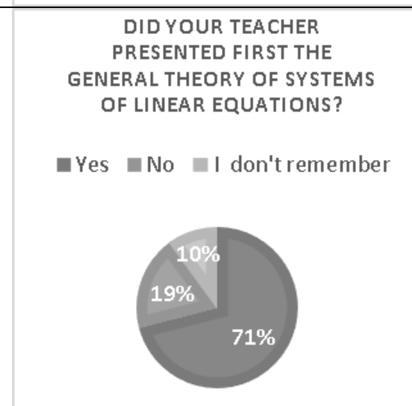
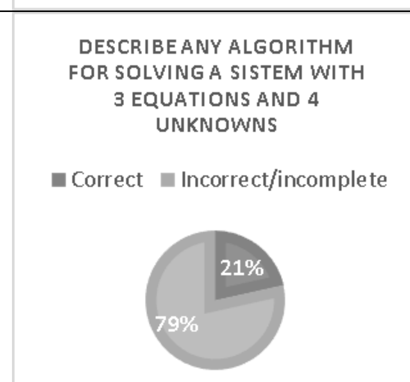
a) transcribe the text of the real problem into mathematical language,
b) solve the mathematical problem find the amount of each type of cheese used for one kilo of pizza and
c) comment on the results obtained.

The main difficulties faced by students are the following: identifying the variables, recognizing the hypothesis for a correct representation of the system, finding the mathematical model for a real-life problem, and the study of the compatibility of the system.

82 could not rewrite the hypothesis as a system of equations. 35 wrote wrong equations and 83 put down a correct form for the system. But only 32 found the right solution.

In our opinion, the results were surprisingly weak. We believe that the word problem was an obstacle to finding the solution. According to Greer (Greer, 1997) students often focus on computing or using algebraic/ arithmetics skills, and not on reformulating the word problems. We think the systems of linear equations are not the most difficult problems in mathematics. We wanted to find out why these results happen, and why the students can not resolve such problems even though they studied the systems (Table 2).

Table 2.

	Questions	Possible answers	Frequency	Circular diagram
1	Did your teacher ask you to solve any real-life problems last year? (Yes/ No)	Yes No	62 138	
2	Have you got examples from your teacher? (None/Few/Many)	None Few Many	30 127 43	
3	Did your teacher present first the general theory of systems of linear equations? (Yes/No/ I don't remember)	Yes No I don't remember	142 38 20	
4	Describe any algorithm for solving a system with 3 equations and 4 unknowns.	Correct Incorrect	43 157	

The role of mathematical modeling in mathematics education is neglected by teachers and from here it follows the weakly results in solving real problems. Consequently, we propose a different way to present the systems of linear equations, based on examples.

4. Conclusions

The examples inspired by real life are very important in teaching mathematics. A good example leads to the success of the didactic approach. The most important and the most difficult activity for the maths teacher is teaching abstract notions. These are considered assimilated by the student if he can use them in new situations. The teacher must choose examples related to real life such as students to focus on interpretation, connections, relations, and not only on computing.

At present, in Romania, regarding the systems of linear equations, most teachers first present the general theory and after give some examples. We consider that by first presenting interesting examples and afterward discovering the applicability of solving systems in the everyday life, the students will be formed into thinking correctly about the problem. Finally, the teacher will present the general notions needed to complete the scientific knowledge. The research can be continued with the application of the mathematical modeling method on a representative sample of high school students, to see to what extent the ability to solve systems changes.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Biembengut M. S. (2016). *Modelagem na Educação Matemática e na Ciência*. Livraria da Física.

Biembengut M.S. (2015). Mathematical modelling, problem solving, project and ethnomathematics: Confluent points. *CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education*, Charles University in Prague, Faculty of Education; ERME, Feb 2015, Prague, Czech Republic. pp.816-820. hal-01287247

Blum, W., & Borromeo Ferri, R. (2009). Mathematical modelling: Can I be taught and learn? *Journal of Mathematical Modeling and Application*, *1*(1), 45–58.

Blum, W., & Leiβ D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling: Education, engineering and economics* (pp. 222–231). Woodhead Publishing Limited.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, application, and links to other subjects-state, trends, and issues in mathematics instruction. *Educational Studies in Mathematics, 22*(1), 37-68.

Boghian, I. (2018). *Methodological Guide for language students and language teachers: English, French, Romanian*. Cluj-Napoca: House of Science Book.

Greer, B. (1997). Modelling reality in mathematics classrooms: The case of word problems. Learning and Instruction, 7(4), 293-307.

Lehrer, R., & Schauble, L. (2003). Origins and evaluation of model-based reasoning in mathematics and science. In R. Lesh, & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 59-70). Mahwah, NJ: Lawrence Erlbaum.

Marcus, S. (1987). *Şocul matematicii* [The shock of mathematics]. Bucharest: Albatros.

Polya G. (1981). *Mathematical discovery: on understanding, learning and teaching problem solving*. New York: John Wiley & Sons.

Possani E., Trigueros M., Preciado J.G., Lozano M.D. (2010). Use of models in the teaching of linear algebra. *Linear Algebra and its Applications, 432*(8), 2125-2140, https://doi.org/10.1016/j.laa.2009.05.004.

Satchwell, R. E., & Loepp, F. L. (2002). Designing and Implementing an Integrated Mathematics, Science, and Technology Curriculum for the Middle School. *Journal of Industrial Teacher Education*, 39(3). Retrieved from https://scholar.lib.vt.edu/ejournals/JITE/v39n3/satchwell.html.

Sriraman, B. (2006). Conceptualizing the model-eliciting perspective of mathematical problem solving. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (CERME 4) (pp. 1686-1695). Sant Feliu de Guíxols, Spain: FUNDEMI IQS, Universitat Ramon Llull..

Swokowski, E., & Cole, J. (2010). *Algebra and Trigonometry with Analitic Geometry*. Classic Twelfth Edition, Cengage Learning.

Haines, C., & Crouch, R. (2007). Mathematical modeling and applications: Ability and competence frameworks. In W. Blum, P. L. Galbraith, H. Henn, & M. Niss (Eds.), Modelling and applications in mathematics education: The 14th ICMI study (pp. 417-424). New York, NY: Springer.

Verschaffel, L., Greer, B., & De Corte, E. (2002). Everyday knowledge and mathematical modeling of school work problems. In K. P. Gravemeijer, R. Lehrer,H. J. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 171-195). Dordrecht, The Netherlands: Kluwer Academic Publishers.