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Reflection in Mathematical Training

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Abstract

In the latest international documents on the development of European education, reflection is addressed as an essential element in the structure of "key competences". The issue of reflection is in line with the development priorities of today's pedagogy of education, in which the ability to self-knowledge, to apply a rational and qualitative analysis of one's intellectual and practical activity is a prerequisite for any purposeful development and cultivation. The article presents a systematic model of the place of reflection and the course of mental processes that precede mastery. The essence of reflective training is explored. An application of the reflective method in training in higher mathematics is presented, in particular, in the study of linear algebra for students of the Technical University.

Key words: Reflection; reflective training; training in mathematics

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1. Introduction

The main areas of our educational system, which is in a state of modernization, are: multilevel system for the training of professionals; orientation of the final result of education to the expansion of education beyond formal education in parallel structures of the continuous educational system; process of informatization of education using modern information technologies, means and methods of informatics for the realization of the program "development of education"; intensification of the learning process at all levels through the introduction of interactive methods and new educational technologies that increase its quality and efficiency. The type of education that stimulates the intellectual and moral development of the learner's personality, activates his potential abilities and trains critical thinking is becoming more and more popular.

In the recent international documents on the development of European education, reflection is considered as an essential element in the structure of "core competencies" needed in the "knowledge society" (Nikolov, 2010). Providing pre-service teachers with face-to-face peer interactions regarding mathematical knowledge and teaching in the classroom is necessary to stimulate deep reflection (Chen, 2015; Çimer, Çimer, & Vekli, 2013). Reflection issues correspond to the development priorities in the modern educational methodology, since the ability of a person to be self-aware and to analyze his mental and practical activity rationally and effectively is a prerequisite for any purposeful development and self-improvement.

2. Theoretical framework

Reflection has a multifunctional character. It is a term that comes from philosophy and is derived from the Latin word reflexio, reflectere, which literally means recollection, reflection. The first mention of reflection is found in ancient Greek philosophy in the works of Heraclitus, Socrates, Plato and Aristotle. Later, experts in various fields of human knowledge became interested in it. Reflection is „a method that can be used to improve professional skills in all areas; it can also help teachers to build professional competence" (van Manen, 1991).

According to **psychologists**, *reflection is a subjective form of psychological existence*. It belongs to the basis of personality formation and is a necessary part of general cognitive abilities. Contemporary Russian psychologists consider it as a form of self-knowledge necessary to gain knowledge about the nature of self-activity and the characteristics of one's personality. From this point of view, *reflection* is used to explain and study consciousness, self-consciousness, and personality.

Philosophers assume that *reflection is a psychological phenomenon*. It enables a person to recognize the existence of a point of view other than his own. Roughly speaking, reflection is the understanding of oneself and others, self and other evaluation, self and other interpretation, etc.

Educators' research aims to determine the influence of various pedagogical factors on the process of reflection formation and development, and to create an innovative environment that stimulates the emergence of various types of innovations.

One of the first Bulgarian authors to deal with reflection is Petar Nikolov. In the mid-1990s, he began to develop reflection as a separate and independent topic, distinguishing between intellectual and personal reflection. As one of the first authors to write about reflection, he argues that the manifestation of reflection can be observed as early as preschool age. Another important contribution of his is the concept of reflective approach in education. Later he developed the idea of reflective choice (Nikolov, 2010, p. 14). He discusses professional pedagogical reflection, which is "an awareness of the achieved level of professional and pedagogical development."

In her didactic-methodological monograph, M. Georgieva (2001) expresses the idea that reflection is not only a feature of theoretical thinking, but can also be considered as a method of forming special skills for learning and developing the student's personality. In general, the author states that for the needs of mathematics education, reflection is one of the most important

mechanisms for stimulating students' mental abilities and cognitive interests. Understanding abstract mathematical material is closely related to reflection. It is an essential means of making sense of understanding that is central to the mastery and application of mathematical knowledge (Figure 1).

The proposed model shows that reflection is related to understanding and comprehension but is not essential for their realization, while, on the other hand, reflection cannot be realized without understanding and comprehension (Aleksiev, 1983). Reflection is central to the acquisition and application of knowledge and skills and forms the basis for the acquisition of reflective skills.

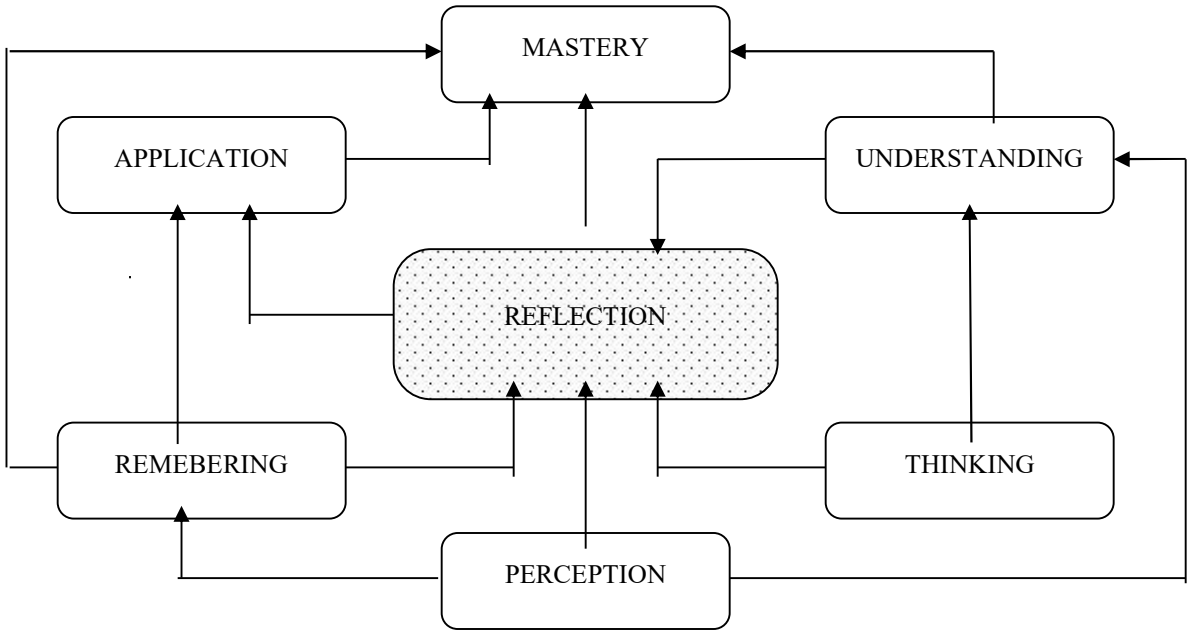


Figure 1. Systematic model for the role of reflection and the sequence of psychological processes that precede mastery

The importance of *reflection* in mastering one's experience is undeniable. In the process of acquiring mathematical experience, reflection is characterized by intense mental activity, purposeful thinking, overcoming problem situations and difficulties in solving challenging mathematical tasks, self-transformation and personality development of the persons involved in this process. In the course of such activity, the learner:

- analyzes the task in order to understand it;
- identifies and retrieves the necessary knowledge;
- applies his/her knowledge in the process of problem solving;
- makes hypotheses;
- applies intellectual agility;
- finds new solutions;
- checks them;
- draws conclusions about his/her own activity;

Based on the literature discussed above, it can be stated that the mere concept of reflection is very broad. "Reflection is a process of self-movement of one's theoretical activity, subjective reflection of phenomena, processes of self-consciousness, self-awareness and behavior of

personality, psyche and internal structure of mentality of the individual... It includes perception, analysis, understanding, reasoning and reflection on one's own behavior, mental states, their peculiarities and patterns, as well as a specific process of self-awareness that reveals the specificity of psychic life" (Desev, 1999, p.49). There are the following types of reflection (Georgieva, 2001): intellectual, praxeological, personal, formal, meaningful, etc.

The relationship "training - reflection" is closely connected with the relationship "training - development". The principle and methodological importance of this relationship in mathematical education is shown (Ganchev, 1999, p. 47):

- in the acquisition of new knowledge under the guidance of a teacher (the acquired knowledge and skills constitute the "closest development zone"); the development of each individual is connected with their precise determination;
- in training, which can contribute more to development than what is contained in its immediate results;
- in teamwork and communication - the driving force of development;
- in internalization and exteriorization as mechanisms of development in mathematical education or, in other words, in the relationship "education-development". A.S. Vîgotski speaks of "acquisition and mastery to a certain degree of accidental recreation and application of more general knowledge and skills not directly acquired through education."

The new information and digital technologies used in mathematics education make it possible to improve the means of reflective teaching for the following reasons:

- the learning process is individualized; individual characteristics are taken into account; learning content is selected to meet the needs of each trainee at each stage, i.e. the computer covers different aspects of the learning/teaching activity;

- optimal combination of individual learning and traditional learning under the guidance of a teacher;

- monitoring of students' performance – they have the opportunity to receive immediate information about the results of their activity.

The methodology for developing skills in the context of teaching mathematics must meet certain requirements related to the formation of students' intellectual and practical reflection. In this regard, the research work of Iv. Ganchev's (1999) research work and his model of the relationship between education and intellectual achievement is of great value. Another important point made by Iv. Ganchev (1999) concerning the relationship "training-reflection" is the mastery of didactic indicator systems:

- to build a didactic system of indicators for each studied term with its definition and at least one sentence providing sufficient conditions for recognition of the objects within its scope;
- to include all theorems studied up to a certain point in the system of indicators of each term, the system providing sufficient conditions for the recognition of the objects from its scope.

The introduction of information technology (IT) makes it possible to use the didactic capabilities of the computer to creatively acquire knowledge and skills and develop the research activity of trainees to stimulate their reflection; "Real conditions are created to provide patterns, generalizations, reflection, goals and meaning within the mass training of "challenge" - all that is inherent in productive types of activities" (Gostev,1985, p.4).

As mentioned earlier, one of the main goals of our educational system is to intensify the learning process at all levels by using interactive methods and new educational technologies. One such modern educational technology is the technology of reflective teaching and its essence consists in the following (Georgieva, 2001):

- setting up a project to activate different psychological processes and functions to the extent of activating reflection beyond students' activity and knowledge;
- restructuring of course materials (in this case mathematics course materials) aimed at reflection beyond knowledge, which in turn enhances learning culture;
- searching for innovative processes related to the development of the subjective experience of the trainees;
- application of active training methods;
- teamwork and individual work;
- development of reflective didactic means, including development of appropriate software containing in itself the activity (certain strategies of cognitive activity);
- implementation of learning and teaching as technological processes;
- increasing the effectiveness of didactic communication;
- integration of extracurricular learning.

An essential feature of reflective education is intellectualization. Intellectualization is considered as a process that promotes the trainees' ability to creatively acquire and apply knowledge and skills, thus positively changing their intellectual potential. The acquisition of knowledge and the formation of skills take place at four levels: reproductive, productive, transfer, and creative. At the first level, the obligatory or optional work creates optimal conditions for the awareness of the decision-making mechanism or the formation of self-regulation skills. On the second level, trainees have the opportunity to act from the position of an active subject. The specifics of productive activities fulfil some of the characteristics of reflection, such as selecting or building a cognitive schema relevant to the experience gained. The third level includes both reproductive and productive knowledge to develop associative, combinatorial, and divergent thinking. "To some extent, this stage allows for timely diagnosis of the trainee's development and even prediction of developmental tendencies" (Georgieva & Cherkezova, 1997). At this stage, the trainee is freed from dependence on a particular cognition scheme and applies one or another combination of appropriate cognition schemes. Characteristic of this stage is that the movement of thought goes beyond the limits of isolated determinacy and reflection correlates with another determinacy. The fourth stage is associated with the highest form of generalization of "proficiency"; with the most undeniable manifestation of reflection as the highest form of management of intellectual processes. Trainees are able to perform heuristic activities in relation to a particular indeterminacy that offers different ways of implementing the reflective experience.

We will show an application of the reflexive approach in higher mathematics education and, in particular, in linear algebra education for students at technical universities. The linear algebra module for students at technical universities involves the study of matrices and determinants and their application to solving practical problems. As classroom experience shows, this part of linear algebra as traditionally taught is difficult for students because of the many new, complex concepts and the interrelationships among them. A complete understanding of the theoretical material is necessary for engineering students for whom higher mathematics is not a core subject to solve practical problems.

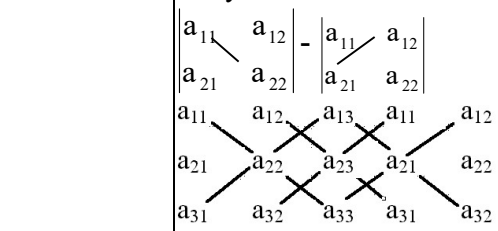
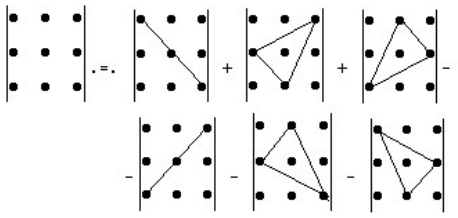
Seminars (classes) are conducted according to the traditional scheme - they begin with reviewing the results of the previous work, studying the new material, solving problems under the guidance of the teacher, individual work, summarizing the results of work, giving homework and checking knowledge. It should be noted that each stage of training involves the inclusion of reflection mechanisms as a result of the introduction of reflection techniques. In this context, the training phases can be presented as follows:

- reflection of the results from previous work;
- visualization of the new material;
- reflective practice under teacher guidance;
- students' reflective individual work;

- seminar reflection;
- self-control of students' knowledge;

Let us consider reflective teaching on the topic "Determinants of second and third order". The description of the seminar is presented in Table 1 (Shatova, Romanovska, 2014).

Table 1. Stages of reflective training (Shatova, Romanovska, 2014)

Stages of reflective training	Stage content	Questions for reflection
Reflection on the results of previous work	Questions for discussion: 1 The concept of "matrix" – definition, designation. 2. Types of matrices. 3. Matrix operations. Properties. 4. Transposed matrix. Inverse matrix.	1. How would you rate your readiness for this course? 2. What previous knowledge do you need to prepare for this course? 3. What difficulties did you have in preparing for the course? Why? How were they overcome? 4. Formulate questions that you were unable to answer when preparing for today's class. 5. Formulate a question about the topic of today's lesson to a colleague that you think is difficult (easy) to answer.
Visualization of the new material	The number $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$ is called **the second-order determinant** The number $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33}$ is called **the third-order determinant** Rules for solving second and third order determinants. I way  II way- as a rule of triangles 	1. Why did we reduce the new material to a scheme? Did you use formulas in the process? 2. Are there advantages to writing down the new material in the form of a schema? What are the advantages of using formulas? 3. What problems did you encounter in describing the material and writing down the formulas? 4. What difficulties do you encounter in reproducing the scheme? Formulas? Why? What are the solutions? 5. Describe the sequence of the schemes. Suggest your own way to present the new material.

	III way - development of the determinant on the elements of the i-th row (k-pillar) $A_{ik} = (-1)^{i+k}.\Delta_{ik} =$ $= (-1)^{i+k}\begin{vmatrix} \dots & \dots & \dots \\ \dots & a_{ik} & \dots \\ \dots & \dots & \dots \end{vmatrix}$ (k — column, i — row) $\Delta = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{ik}A_{ik} + \dots + a_{in}A_{in} = (-1)^{i+1}a_{i1}\Delta_{i1} + (-1)^{i+2}a_{i2}\Delta_{i2} + \dots + (-1)^{i+k}a_{ik}\Delta_{ik} + \dots + (-1)^{i+n}a_{in}\Delta_{in}$	
Reflective practice under the guidance of the teacher	1. Discussion questions: a) Define the second and third order determinant. б) Define the algebraic complement в) What are some ways to calculate determinants? г)Name the properties of determinants 2. Ask questions to discuss and select rational methods for calculating determinants. 3. Solving mathematical problems **a)** $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix}$ **b)** $\begin{vmatrix} 3 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix}$	1. What types of problems have we solved? 2. Formulate the text and the purpose of the problems. 3. Formulate the problem-solving algorithm. 4. What difficulties did you encounter in solving the problem? Why? How can they be overcome? 5. Formulate the difficulties you have encountered. 6. What problems can be encountered in solving such mathematical problems? 7. What remained unclear to you in solving the problems? 8. What other ways are there to solve these tasks? Can you suggest any? 9. Who among your classmates do you think gave the best answers to the questions asked? And why?
Reflection of the learning activities during the seminar	1. Correspondence between the objectives set at the beginning of the class and the results obtained at the end of the class. 2. Homework. 3. Summary of theoretical information on "second and third order determinants and their properties". 4. Solving mathematical problems. a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$ **b)** $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix}$ **c)**	1. Formulate the purpose of today's seminar. 2. Did we reach the goal of the seminar? 3. what did you learn in this seminar? 4) Formulate the problems you encountered during the seminar. 5. Which learning material could you cope with the best? And what could you not cope with? 6. At what point during the seminar did you feel most successful? Least successful? 7. What would you brag about during the seminar class? 8. If you had the opportunity, what

	$\begin{vmatrix} -5 & 1 & 2 \\ 3 & 0 & -6 \\ -2 & 7 & -3 \end{vmatrix}$	would you do differently during this seminar lesson? 9. What would you change about the teacher's activity in this seminar group? 10. What did you like best about this seminar lesson? What did you not like about it? 11. How would you evaluate your work in this seminar? 12. What questions would you like answered after this seminar? 13. Which of the assigned homework could be difficult for you to complete? 14. Which fragments of the current topic should you pay more attention to?
Students' individual reflective activity	Solve the following test: 1. The determinant $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix}$ is equal to: а) 0 б) -5 в) 23 г) 130 2. The determinant $\begin{vmatrix} 1 & -4 & 3 \\ 2 & -1 & 0 \\ 1 & 2 & 4 \end{vmatrix}$ is equal to: а) 0 б) -51 в) 13 г) 43 3.The algebraic addition (the adjunct quantity) of the element a_{23} of the determinant $\begin{vmatrix} 1 & -4 & 3 \\ 2 & -1 & 0 \\ 1 & 2 & 4 \end{vmatrix}$ is equal to: а) -6 б) 0 в) 6 г) 13	1. How would you grade your performance on the exam? 2. Does your own grade agree with the teacher's grade? If not, why? 3. Which of the tasks on the test was the most difficult for you? The easiest? 4. What prevented you from completing some of the tasks? 5. Which of your knowledge proved insufficient to solve all the tasks in the test? 6. Formulate the problems you encountered in completing the tasks in the test. What options do you have to overcome these difficulties? 7. What possibilities do you have to fill the missing gaps in your knowledge?

We can say that the application of certain methods of organizing reflective activities makes it possible to create a reflective environment during seminar teaching, in which students take a reflective position under the guidance of the teacher. The described method of organizing reflection activities in linear algebra courses not only improves students' knowledge of the subject, but also develops reflection skills that can be applied to other higher mathematics courses.

Reflection is used to intentionally connect the cognitive process to the application of knowledge in practice. Reflection is a process - an intellectual process associated with the

intellectual development of the learner. In education it is a driving force for the formation of his personality. Reflection is expressed in discussions, conversations, etc. When working with mathematical problems, the reflection process is linked to the solution of concrete practical problems. It refers to the subject's appropriate selection of a suitable idea or method or a combination of methods, to the correct selection of the necessary theoretical knowledge for solving a particular task. In coping with mathematical experiences, reflection is characterized by: intense mental activity, purposefulness, overcoming difficulties in solving tasks, development and self-transformation of personality. One of the most important features of the reflective learning process reduces to such qualitative self-transformation of the learner, which makes him an active subject of his own learning activity, where the object of learning is not so much the acquisition of mathematical knowledge and skills, but the activity of the learner himself (Boykina & Todorova, 2020).

3. Conclusion

In summary, the following conclusions can be drawn. Learning should be self-conscious or reflexive learning. The basic idea of reflective learning is that the intellectual-creative freedom of the teacher is really possible only under the conditions of sufficient professional autonomy and freedom to make important decisions in his work.

The need to study reflection from different aspects and with different approaches is related to improving the quality of teaching and learning, especially mathematical learning. Understanding the nature and mechanisms of intellectual and praxeological reflection will ensure the provision of "evolving education" and facilitate the transition from an extensive to an intensive developmental path. The intelligent management of mathematics learning as a multimedia learning project with a fundamentally new approach and system on the Internet, individual learning in international programs, interactive lectures abroad and others are the way to reflective learning.

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