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Reflection in Mathematical Education

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Abstract

In the latest international documents on the development of European education, reflection is mentioned as an important element in the structure of "key competences". The issue of reflection is in line with the development priorities of today's pedagogy of education, in which the ability of self-knowledge, the application of rational and qualitative analysis of one's intellectual and practical activity is a prerequisite for any purposeful development and cultivation. The article presents a systematic model of the place of reflection and the sequence of mental processes that precede mastery. The nature of reflection training is explored. An application of the reflection method in higher mathematics education is presented, in particular, in the study of linear algebra for students of Technical University.

Key words: Education in mathematics; reflection; reflective training

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1. Introduction

The main areas of our educational system, which is in a state of modernization, are: a multilevel system for the training of specialists; the orientation of the final result of education to the expansion of education beyond formal education in parallel structures of the continuous educational system; the process of informatization of education using modern information technologies, means and methods of informatics for the implementation of the program "Development of education"; the intensification of the learning process at all levels through the introduction of interactive methods and new educational technologies that increase its quality and efficiency. The type of education that stimulates the intellectual and moral development of the learner's personality, activates his potential abilities and trains critical thinking is becoming increasingly popular.

In the recent international documents on the development of European education, reflection is considered as an essential element in the structure of "core competences" needed in the "knowledge society". Reflection issues correspond to the development priorities in the modern educational methodology, as the ability of a person to be self-aware and analyse his intellectual and practical activity rationally and effectively is a prerequisite for any purposeful development and self-improvement.

Reflection has a multifunctional character. It is a concept that originated in philosophy and is derived from the Latin word reflexio, reflectere, which literally means "to think back", "to reason". Reflection is first mentioned in ancient Greek philosophy in the works of Heraclitus, Socrates, Platon and Aristotel. Later, experts in various fields of human knowledge became interested in it. According to psychologists, reflection is a subjective form of psychological existence. It is the basis of personality formation and a necessary component of general cognitive abilities. Contemporary Russian psychologists consider it as a form of self-knowledge, which is necessary to gain knowledge about the nature of self-activity and the characteristics of one's personality. From this point of view, reflection is used to explain and study consciousness, self-consciousness and personality. Philosophers assume that reflection is a psychological phenomenon. It enables a person to recognize the existence of a point of view other than his own. Roughly speaking, reflection is understanding oneself and understanding others, self-assessment and assessing others, self-interpretation and interpreting others, etc.

2. The systematic model of reflection

The research of educators aims to determine the influence of various pedagogical factors on the process of reflection formation and development, and to create an innovative environment that stimulates the emergence of different types of innovations. One of the first authors in Bulgaria to deal with reflection is Petar Nikolov. In the mid-1990s, he began to develop reflection as a separate and independent topic, distinguishing between intellectual and personal reflection. As one of the first authors to write about reflection, he shares the idea that the manifestation of reflection can be observed as early as preschool age. Another important contribution of his is the concept of reflective approach in education. Later he developed the idea of reflective choice. Nikolov (2010, p. 14) discusses the professional pedagogical reflection, which is "an awareness of the achieved level of professional and pedagogical development."

In her didactic-methodological monograph, M. Georgieva (2001) expresses the idea that reflection is not only a feature of theoretical thinking, but can also be considered as a method of forming special skills for learning and developing the student's personality. In general, the author states that for the needs of mathematics education, reflection is one of the most important mechanisms to develop students' intellectual abilities and cognitive interests. Understanding abstract mathematical material is closely related to reflection. It is an essential means of making sense of understanding, which is at the heart of mastering and applying mathematical knowledge (Figure1). The proposed model shows that reflection is related to understanding and comprehension but is not essential for their realization, while on the other hand reflection cannot be realized without understanding and comprehension (Aleksiev,1983). Reflection is the heart of the acquisition and application of knowledge and skills, a basis for the acquisition of reflective skills.

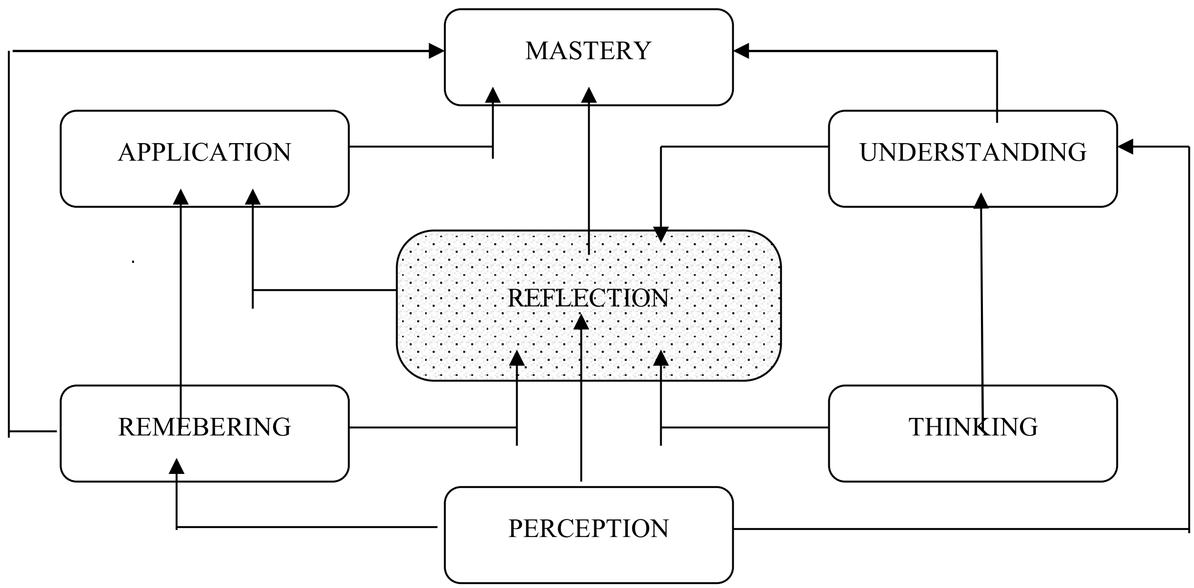


Figure1. Systematic model for the role of reflection and the sequence of psychological processes that precede mastery

The importance of reflection for the mastery of one's experience is undeniable. In the process of acquiring mathematical experience, reflection is characterized by intense mental activity, purposeful thinking, overcoming problem situations and difficulties in solving challenging mathematical tasks, self-transformation and personality development of the persons involved in this process. In the course of such activity, the learner:

- analyses the task in order to understand it;
- identifies and retrieves the necessary knowledge;
- applies his/her knowledge in the process of problem solving;
- makes hypothesis;
- applies intellectual agility;
- finds new solutions;
- checks them;
- draws conclusions about his/her own activity.

Based on the literature review discussed above, it can be concluded that the mere concept of reflection is very rich. "Reflection is a process of self-movement of one's theoretical activity, subjective reflection of phenomena, processes of self-consciousness, self-consciousness of personality and behaviour, psyche and internal structure of mentality of the individual... It includes perception, analysis, understanding, reasoning and reflection on one's own behaviour, mental states, their peculiarities and patterns, as well as a specific process of self-knowledge that reveals the specificity of mental life." (Desev, 1999, p. 49).

There are the following types of reflection: intellectual, praxeological, personal, formal, meaningful, etc. (Georgieva, 2001). The relationship "training - reflection" is closely connected with the relationship "training - development". The principle and methodological importance of this relationship in mathematical education manifests itself:

- in the acquisition of new knowledge under the guidance of a teacher (the acquired knowledge and skills constitute the "narrowest zone of development"); the development of each individual is related to their precise determination;
- in training that can bring more in terms of development than what is contained in its immediate results;
- in teamwork and communication - the driving force of development;
- in internalization and externalization as mechanisms of development in mathematical education or, in other words, in the relationship "education-development".

A.S. Vigotski speaks of "acquisition and mastery to some extent of accidental recreation and application of more general knowledge and skills that do not come directly from education" (Ganchev, 1999, p. 47). The new information and digital technologies used in mathematics education allow to improve the means of reflective teaching for the following reasons:

- the learning process is individualized; individual characteristics are taken into account; training effects are selected that meet the needs of each trainee for the given stage, i.e. the computer covers different aspects of learning/teaching activity;

- optimal combination of individual learning and traditional learning under the guidance of a teacher;

- monitoring of students' performance - they have the opportunity to receive immediate information about the results of their activity.

The methodology of skill development in mathematics teaching must meet certain requirements related to the formation of intellectual and practical reflection of students. In this respect, Iv. Ganchev's research and his model of the relationship between training and intellectual performance is valuable (Ganchev, 1999). Another important point made by Iv. Ganchev addresses in relation to the "training-reflection" relationship is the mastery of didactic indicator systems:

- for each studied concept a definition and at least one theorem that provides sufficient conditions for the recognition of objects within its scope; to build a didactic system of indicators;
- to include all the conditions of the theorems studied up to a certain point in the system of indicators of each concept, the system providing sufficient conditions for the recognition of the objects from its scope.

3. The technology of reflective teaching

The introduction of IT makes it possible to use the didactic capabilities of the computer for the creative acquisition of knowledge and skills and to develop the research activity of learners to stimulate their reflection. "Real conditions are created for providing patterns, generalizations, reflection, goals and meaning within the mass training of "challenge" - all that is inherent in productive types of activity" (Gostev,1985, p. 4). As previously noted, one of the main goals of our educational system is intensification of the learning process at all levels by implementing interactive methods and new educational technologies. Such modern educational technology is the *technology of reflective teaching* and its essence consists in the following:

- setting up a project to activate different psychological processes and functions to the extent of activating reflection beyond students' activity and knowledge;
- restructuring of course materials (in this case mathematics course materials) aimed at reflection beyond knowledge, which in turn enhances learning culture;
- searching for innovative processes related to the development of the subjective experience of the trainee;
- application of active methods of training;
- teamwork and individual work;
- development of reflective didactic means, including development of appropriate software containing in itself the activity (certain strategies of cognitive activity);
- implementation of learning and teaching as technological processes;
- increasing the effectiveness of didactic communication;
- integration of extracurricular learning (Georgieva, 2001).

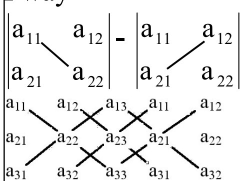
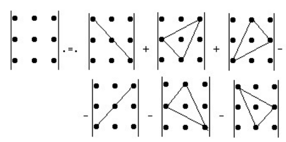
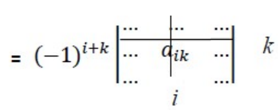
An essential feature of reflective training is intellectualization. Intellectualization is viewed as a process related to enhancing the ability of trainees to creatively acquire and apply knowledge and skills and thus to alter positively their intellectual potential. Acquisition of knowledge and formation of skills take place at *four levels*: *reproductive, productive, transfer and creative.* At the first level, the obligatory or optional work creates optimal conditions for the awareness of the decision-making mechanism or the formation of self-regulation skills. At the second level, trainees have the opportunity to act from the position of an active subject. The specifics of productive activities fulfill some of the characteristics of reflection, such as the selection or construction of a cognitive schema relevant to the experience gained. The third level includes both reproductive and productive knowledge to develop associative, combinatorial and divergent thinking. "This level to some extent enables timely diagnosis of the trainee's development and even predicts tendencies in his development" (Georgieva, Cherkezova,1997). At this level, the trainee is freed from dependence on a particular cognitive scheme and applies one or another combination of appropriate cognitive schemes. Characteristic of this level is that the movement of thought goes beyond the limits of isolated determinacy and relates reflection to another determinacy. The fourth level is associated with the highest form of generalization of "ability"; with the most undeniable manifestation of reflection as the highest form of management of intellectual processes. Trainees are able to perform heuristic activities in the context of a certain indeterminacy that offers different ways of implementing the reflective experience. We will show an application of the reflective approach in higher mathematics education and, in particular, in linear algebra education for students at technical universities.

The Linear Algebra module for students at technical universities involves the study of matrices and determinants and their application to solving practical problems. As the classroom experience shows, this part of linear algebra as traditionally taught is difficult for students because of the many new complex concepts and the interrelationships between them. A complete understanding of the theoretical material is necessary for students of engineering courses for which higher mathematics is not a core subject to solve practical problems. Seminars (classes) are conducted according to the traditional scheme - they begin with reviewing the results of the previous work, studying the new material, solving problems under the teacher's guidance, individual work, summarizing the results of work, giving homework and checking knowledge. It should be noted that each stage of training involves the inclusion of reflection mechanisms as a result of the introduction of reflection techniques. In this context, the phases of training can be presented as follows:

- reflection of the results from previous work;
- visualization of the new material;
- reflective practice under the guidance of the teacher;
- students' reflective individual work;
- seminar reflection;
- self-control of students' knowledge.

Let us consider the reflective teaching on the topic "Determinants of the second and third order". The description of the seminar is presented in Table 1 (Shatova, Romanovska, 2014).

Stages of reflective training	Stage content	Questions for reflection
Reflection on the results of previous work	Questions for discussion: 1 The concept of "matrix" – definition, designation. 2. Types of matrices. 3. Matrix operations. Properties. 4. Transposed matrix. Inverse matrix.	1. How do you evaluate your readiness for this class? 2. What kind of previously obtained knowledge do you need in order to prepare for this class? 3 What difficulties did you have in preparing for the class? Why? How were they eliminated? 4. Formulate questions you couldn't answer when preparing for today's class. 5. Formulate a question about the topic of today's class for a colleague you think it would be difficult (easy) to answer.
Visualization of the new material	The number $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$ is called the second-order determinant	1. Why did we reduce the new material to a scheme? Did you use formulae in the process? 2. Are there any advantages to writing down the new material in the form of a scheme?

	The number $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33}$ is called **the third-order determinant** Rules for solving second and third order determinants. I way  II way- as a rule of triangles  III way - development of the determinant on the elements of the i-th row (k-pillar) $A_{ik} = (-1)^{i+k}.\Delta_{ik} =$  $\Delta = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{ik}A_{ik} + \ldots + a_{in}A_{in} = (-1)^{i+1}a_{i1}\Delta_{i1} + (-1)^{i+2}a_{i2}\Delta_{i2} + \ldots + (-1)^{i+k} a_{ik} \Delta_{ik} + \ldots + (-1)^{i+n}a_{in}\Delta_{in}$	What are the advantages of using formulae? 3. What problems did you encounter when describing the material and writing down the formulae? 4. What difficulties do you have when reproducing the scheme? Formulae? Why? What are the solutions? 5. Describe the sequence of schemes. Suggest your own way of presenting the new material.
Reflective practice under the guidance of the teacher	1. Questions for discussion: a) Define second and third order determinant б)Define algebraic complement в) What ways do you know about calculating determinants? г) name the properties of determinants 2. Questions for discussing and choosing rational methods for calculating	1. What types of problems did we solve? 2. Formulate the text and purpose of the problems. 3. Formulate the problem solving algorithm. 4. What difficulties did you encounter in the process of problem solving? Why? How can they be overcome? 5. Formulate the difficulties you encountered. 6. What problems can arise when solving such math problems? 7. What remained unclear to you in the solution of the problems?

	determinants. 3. Solving mathematical problems a) $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix}$ b) $\begin{vmatrix} 3 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix}$	8. What other ways are there to solve these problems? Can you suggest any? 9. Who from your fellow students do you think gave the best answers to the questions set? Why?
Reflection of the learning activities during the seminar	1. Correspondence between the goals set at the beginning of the class and the results obtained at the end. 2. Homework assignment. 3. Summarizing theoretical information on "Second and third order determinants and their properties". 4. Solving mathematical problems. a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$ b) $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix}$ c) $\begin{vmatrix} -5 & 1 & 2 \\ 3 & 0 & -6 \\ -2 & 7 & -3 \end{vmatrix}$	1. Formulate the purpose of today's seminar. 2. Did we accomplish the goal of the seminar? 3. What did you learn at this seminar? 4. Formulate the problems you encountered during the seminar. 5. Which learning material did you manage to master best? And what you did you not manage to master. 6. At what point during the seminar did you feel most successful? Least successful? 7. What would you boast about during the seminar class? 8. Given the chance, what would you do differently at this seminar class? 9. What would you change in the teacher's activity at this seminar class? 10. What did you like best about this seminar class? What didn't appeal to you? 11. How would you assess your work at this seminar class? 12. What questions would you like to have answered after this seminar class? 13. Which of the assigned homework tasks may be difficult for you to accomplish? 14. Which fragments of the current topic should you pay more attention to?

Students' individual reflective activity	Solve the following test: 1. The determinant $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix}$ is equal to: а) 0 б) -5 в) 23 г) 130 2. The determinant $\begin{vmatrix} 1 & -4 & 3 \\ 2 & -1 & 0 \\ 1 & 2 & 4 \end{vmatrix}$ is equal to: а) 0 б) -51 в) 13 г) 43 3.The algebraic addition (the adjunct quantity) of the element a_{23} of the determinant $\begin{vmatrix} 1 & -4 & 3 \\ 2 & -1 & 0 \\ 1 & 2 & 4 \end{vmatrix}$ is equal to: а) -6 б) 0 в) 6 г) 13	1. How would you grade your performance at the test? 2. Is your own test grade the same as the teacher's grade? If not, why? 3. Which of the problems in the test proved most difficult for you? The easiest? 4. What prevented you from solving some of the problems? 5. Which of your knowledge proved to be insufficient for solving all the problems in the test? 6. Formulate the problems you encountered while accomplishing the tasks in the test. What are your ways for overcoming these difficulties? 7. What are your ways for filling in the gaps of lacking pieces of knowledge?

We can say that the use of certain methods for organizing reflective activities makes it possible to create a reflective environment during seminar classes, in which students take a reflective position under the guidance of the teacher. The described method of organizing reflective activities in linear algebra classes not only improves students' knowledge of the subject, but also develops reflective skills that can be transferred to other higher mathematics classes.

Conclusions

From this the following conclusions may be drawn: a) the need to study reflection from different aspects and with different approaches is related to improving the quality of teaching and learning and, in particular, mathematical education; b) understanding the nature and mechanisms of intellectual and praxeological reflection will ensure the provision of "developing education" and facilitate the transition from an extensive to an intensive developmental path.

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