

A HISTORICAL RESEARCH ON THE DIDACTICS OF MATHEMATICS

Costică LUPU^{a*}

^a“Vasile Alecsandri” University of Bacău, Romania

Abstract

The relevance of psychological learning theories for establishing a model of science didactics is substantial. Because e-learning theories are psychological theories, when it comes to transforming them into training models applicable to science didactics framework the challenge is content and method related. Psychological theories are based on training models and have to be known including as theories of student personality development. There are psychological descriptive and prescriptive theories which tell us the immediate consequences or implications of an act or fact. For example, most children do not have the notion of reversibility before seven years of age. Instead, theories of instruction based on models of instruction have two pedagogical qualities that are relevant also for science didactics: - it is prescriptive, meaning that it sets out rules to efficiently achieve a certain level of knowledge or skills and offers a unit for critical appreciation or evaluation irrespective of the particular way of teaching-learning; - it is normative, because it sets criteria and proclaims conditions for achieving them, criteria that need to have another level of generality.

Key words: discipline didactics, historical approach, mathematical education, psychological theories

Introduction

Science didactics should not, for example, specify in concrete manner the conditions for learning third grade Arithmetics, as this may be deducted from a more general view on mathematical education, a conception which actually needs to develop the essence of the Didactics of Mathematics. Based on these considerations, methodologically speaking, our task is twofold: a) to identify the most important psychological theories that may support us in building a paradigm

* Assoc. Prof. PhD

E-mail address: costica_lupu@ub.ro , costica_lupu@yahoo.com

of the Didactics of Mathematics; b) to approach these psychological theories as possible models for sufficient training, namely from the perspective of what I. Neacșu (1999, pp.73-92) calls the theory of the conversion of learning into training models.

Classification of psychological theories of learning

We shall first and foremost draw a classification of the psychological theories of learning, based on the existing specialized literature. We shall here refer to a famous treaty on learning theories that presents, among a long list, the following learning theories: theories based on classical and operant conditioning (Pavlov); behaviourism theories (P. Tolman); theories based on functionalism (Holl and Catell); theories based on psychoanalysis (Freud) etc. (apud Miclea, 1999). Two American authors, J. R. Davidtz and S. Ball (1981), have proposed a pyramidal hierarchy of learning theories, represented on six levels:

1. Learning by: a) classical conditioning (Pavlov); b) association of stimuli and responses (Guthrie); c) by setting/instrumental conditioning (Thorndike, Skinner); d) by imitation of social and moral behaviour (Bandura);
2. Learning by: a) generalization (Kendell); b) discrimination (Backer);
3. Learning the concepts by clarification of stimuli with common features (Piaget);
4. Learning principles by hierarchical juxtaposition of concepts (Gagne-Bruner);
5. Learning by observing how a problem is solved by the teacher or by the student's solving the problem himself (Dewey, Ausubel);
6. Learning the behaviour by solving complex, contradictory problems (Ausubel).

S. Cristea (2014) proposes a classification of learning theories with a view to a better exploitation of these as potential models of education applied at all levels of the learning process. The classification of learning theories that are useful from the perspective of the Didactics of Mathematics comprises:

1. Theories based on conditioning learning schemes (see the conditioning hierarchical-cumulative learning theory-R. M. Gagne);
2. Theories based on building cognitive structures of learning (J. Piaget's model of building cognitive structures of learning/education; L. S. Vygotsky's model of building social-cultural

structures of learning/education; J. S. Bruner's social-pedagogical cognitive model of building learning/education).

For building the paradigm of the Didactics of Mathematics as a science of teaching, we considered it appropriate to analyse the psychological constructivist (J. Piaget) and social-cultural (Bruner, Ausubel) theories (Lupu, 2008). We shall interpret the psychological theories of J. Piaget as a possible model of genetic structure of education. The model exploits the stages of cognitive development in connection with genetic evolution. The primary school interval is concerned with the stage of concrete operations (6-7 to 11-12 years) and that of formal operations (11-12 to adult). The Didactics of Mathematics needs to propose structures which match the psychological areas specific to the two stages: acquisition of reversible thinking, clarification of actions, writing and solving of logical problems supported by concrete, perceptible materials; solving problems without the support of objects, using evaluation reports, conceptual models close to those of an adult.

Activity objectives according to the genetic evolution of thought operations

Mathematics course objectives will be proposed according to the quality of genetically determined thinking operations. We should also consider other concepts of Piaget (1972):

- *adaptation* - learning mathematical concepts should be based on a balance between what the student assimilates and possibilities of environmental adaptation (how much he learns in class, at home). School adjustment takes place if there exists an accommodation with the reality of the school task set by the teacher.

- *intelligence* – the cognitive resource of intellectual adaptation which supports the process of adaptation by optimizing the relation between assimilation and accommodation.

- *function* - the cognitive objects proposed by the Maths teacher should imply the involvement of students in content assimilation, adaptation to the environment and learning organization.

J. Piaget (1972) is especially preoccupied with founding a useful pedagogic solution for the paradigm of the Didactics of science. The great Swiss psychologist speaks about “assimilating the real by action and transformation”. For him, to know is to assimilate the real structures and transform the structures, thus intelligence becomes an indirect extension of action. The cognitive structures proposed for the study of Mathematics should stimulate the child's thinking as a resource of reorganization of reality through schemes and action. Piaget proposed the same

schemes of structuring contents in primary, middle and secondary education. In primary education, 7-11 years, the concrete operative intelligence is manifested in relation to objectives, not to assumptions. Mathematics needs didactic concrete objectives. In middle education, operational thinking becomes formal, supporting independent, authentic mathematical thinking. For example, the capacity of combinatorial action, reversing operations based on the principles of reversal and reciprocity etc. J. Piaget performed a special analysis of the Didactics of Mathematics. His paradoxical finding was that although the discipline of Mathematics builds a direct logic reasoning, it is hard to conceive that some well-equipped subjects able to use mathematical logical structures spontaneously are handicapped in understanding the structures resulting from these logical structures.

Solving this paradox constitutes the fundamental problem of the Didactics of Mathematics. On the one hand, the solution should be the expression of pedagogical optimism, because it benefits from the logical mathematical structures with which the student is born. On the other hand, it should comprise learning models based on operational structures of intelligence, which are of a logical-mathematical nature. Firstly, the Didactics of Mathematics should enable the child to solve problems without effort, just like when singing well without knowing the theory of the solfeggio or without reading the score. Teaching Mathematics should stimulate the child to reflect on his own logical-mathematical structures. The technical language used in this respect should not come into conflict with the spontaneous logical-mathematical structures. Secondly, all learning structures should follow a deductive pattern, which offers the advantage of their success and the diminished risk of failure when the student does not meet a certain scheme.

The conclusion formulated by J. Piaget (1972) is the premise for the development of a paradigm of the Didactics of Mathematics as a Didactics of Science. The central problem of teaching Mathematics is that of mutual adjustment between one's spontaneous structures of intelligence and the schedule or methods for Mathematical areas of learning. The proposed solutions, based Bourbaki's study on school work, rely on content modernization and the exploitation of the genetic constructivism of intellectual operations. Pedagogically, the fundamental problem is that "of finding the most appropriate methods to support the transition from natural but irrational structures to the reasoning of some mathematical structures and transposing them into theory (Piaget, 1972, p.17). Solving this issue of substance of the Didactics of Mathematics will enable the pedagogical valuing of the social cognitive conflict inherent in Mathematics, namely the conflict between the "operational handling of structures and the symbolic language that allows their expression". The methodological solutions proposed by J.

Piaget, defined by him as general methods that may be applied to Mathematics, are also interesting:

1. methods of reception or transmission of knowledge – although criticized, these are useful if they stir the student's interest, causing action and generating a kind of conflict between private cases and general lines;

2. active methods leading to action not only at the verbal level but also at the level of thought, valuing the spontaneity of students;

3. intuitive methods that activate the connection between concrete and abstract operations, being stimulated by the modernization of audio-visual equipment: movies, television, videos, computer;

4. scheduled methods: Piaget refers to the advantages of programmatic training.

Another psychological behavioural theory for building a model of the Didactics of Mathematics was elaborated by the great Russian psychologist L.S. Vygotsky (1972). Vygotsky's theory differs from that of Piaget, which is framed within the genetic constructivism category of a psychological order. Vygotsky is a representative of social cultural construction which will influence the work of J. S. Bruner, the foundation of the modern and postmodern curriculum after the 60s and 70s. Unlike Piaget, Vygotsky considers that a child's cognitive development is not genetic, but is influenced by the child's cultural environment, relationships with adults (parents, friends, teachers). Vygotsky believes that development may be anticipated and stimulated based on a quality adult-child interaction. The Didactics of Mathematics should take into account two concepts proposed by Vygotsky, in order to stimulate the development of the child's cognitive structures:

- the concept of scaffolding that defines the social cultural frame built by the adult for which the child acquires mechanisms of thinking and learning which further support his understanding of the world;

- the concept of zone of proximal development, which is essential for the teacher because it defines the area between the actual level of the child and the child's potential level.

We will underline the pedagogical implications of Vygotsky's theory, meanings which allow its transformation into a training model:

- underlining the importance of conscious social learning in the context of interactions with adults;

- the importance of the framework/scaffolding in which learning occurs, which proposes teaching situations based on consolidation, organization, control;
- the importance of collaboration with adults, which makes learning to be more useful, also leading to the development of language;
- the importance of the teacher as a mediator that is also familiar with the child's zone of development (the discovery of this zone offers the chance of maximum success);
- the importance of didactic technology which needs to be evaluated according to the contribution to learning in a concrete environment;
- the importance of the natural acquisition model (proposed by the psychologist Binette) combined with the social-cultural structural model of learning which value the natural/biological functions and the higher psychological functions (thinking);
- the importance of the genetic law of cultural development of a child which expresses the cultural sociogenesis of learning - each function appears on two levels: social and psychological; the higher mental functions depend on education/training and the social environment (logical memory, thinking, will, attention);
- the importance of the complex development of the child reflects the interdependence between the acquired cultural experience, stage of development, the quality of the learning methods used (including the adult initiative);
- the importance of the properties of scientific materials in the context of school training through active learning based on three formative values, which are criteria for assessing the quality of training: generalization of notions, awareness of notions, systematization of notions.

Bruner's theory (apud Miclea, 1999) was strongly influenced by Vygotsky's research, the American psychologist developing and applying his ideas in education. Bruner elaborated a model of cognitive structures of learning which, he believes, ensures the key to success. Like Vygotsky, he regards as important the social cultural and pedagogic context where learning takes place, hence his insistence on the content of a cognitive structure which matches the student's learning style in concrete situations.

So, the Didactics of a discipline established as applied didactics needs to approach differently the presentation of the studied object, as shown in the culture of the society. The author refers to three ways:

a) the “inactive way” – meaning representation of the material predominantly by action (see the Didactics of Mathematics in preschool and primary education);

b) the “iconic way” – meaning by images (see the Didactics of Mathematics in secondary education, based on using schemes);

c) the “symbolic way by different forms of language” (secondary education which proposes mathematical language internalization).

The Didactics of the discipline should take into account three social factors which amplify learning, and psychological elements of the person who resorts to action, image, symbol. The pedagogical problem which needs to be tackled by the Didactics of Mathematics is the learning of mathematical contents, namely symbolic learning (by abstract language) and iconic learning which helps students.

Mathematics constitutes the content upon which the discipline of Didactics exerts its methods. It adjusts and becomes particular to this content. The main tasks of the Didactics of the discipline of Mathematics are:

1) The model of discipline didactics should select, from the science of Mathematics, the concepts, results and fundamental ideas which will be taught to students. The selection is done based on some general principles, which take into account: - the study of the development of Mathematics and its perspectives; - the laws established by psychology; - the most efficient strategies indicated by pedagogy.

2) The selection of the knowledge that will be passed to students is organized according to certain levels of attractiveness and degrees of rigor and complexity.

3) The model of discipline didactics should identify the main traits, tools, methods and applications, characteristic of the different Mathematical disciplines, and should indicate the patterns of Mathematical thinking accessible to students at different ages. According to these three tasks, each country establishes its own curriculum for Mathematics and the corresponding school syllabi.

4) The content and methods of elementary Mathematics are efficient tools in developing the ability to abstract and generalize, in the development of the students’ creativity, perseverance and will.

5) Mathematics resulted from man’s effort to adapt to the surrounding physical environment. At the same time, the surrounding environment became more easily understood with the help of

certain Mathematical models. That is why the process of teaching-learning-evaluation should take into account these considerations.

6) Mathematics should also be correlated with other disciplines studied in school, because discipline didactics has the task to investigate the way in which Mathematical knowledge becomes useful to other disciplines.

7) The most important task in terms of the quantity of discipline Didactics is the detailed methodological description of each theme of study, indicating the appropriate ways to explain it in as an accessible way as possible.

8) Establishing the specific control tools for the students' activity and the specific evaluation tools for the learning progress is a task of discipline Didactics.

9) How individual study is organized, in terms of using textbooks, mathematical journals, collections of problems is also the task of discipline Didactics. Also, discipline Didactics should deal with the organization of activities outside the classroom, such as Mathematical circles, local, national mathematical contests.

By deeply and minutely analysing the teaching-learning-evaluation of Mathematics, the Didactics of the discipline of Mathematics establishes guidelines for the organization of this process. It should provide appropriate answers to the variety of educational situations found in practice. The results of the research from applied Didactics and science Didactics are used and adapted by teachers to the various concrete situations. These should also add new data based on their didactic experience.

Analysis of handbooks and syllabi of Mathematics valid in the educational system of various countries

To understand the evolution of Mathematical education, a historical excursion should be conducted into the preoccupation to teach Mathematics in the course of history, and an analysis of the textbooks, syllabi and Didactics applied to Mathematics in other countries should be performed. There follows an analysis of the framework for realizing the systems during the time intervals of 1950-1989 and 1990-2014. The research relies on the study of documents (school syllabi and methodologies), the application of the historical approach through a synchronic (static, in relation to the respective moment) and diachronic (looking at things in their evolution) approach. There is a presentation of the structure as well as of the author's motivation, the

commentary having both a synchronic (from the perspective of traditional Didactics) and a diachronic perspective in current terms, from the perspective of the model of applied Didactics, as well as that of the model of science Didactics). Throughout history, all peoples have been concerned with selecting the contents to be passed down to students, as well as with organizing them according to certain levels of attractiveness and degrees of rigor and complexity, in an attempt to use the indications of pedagogy efficiently. The term of “Mathematics methodology” was proposed by F. A. Diesterveg (1790-1866) in 1836, in an era of pedagogical burst, following the activity and works of several pedagogues such as I. A. Comenius (1592-1670) and I. G. Pestalozzi (1746-1827), the latter author of the work “Intuitive Studying number”.

We may say that concerns for transmitting Mathematics have existed ever since Mathematics itself occurred. Thus, we have the two Egyptian papyrus books (the Rhind papyrus from London and the papyrus from Moscow) which represent some sort of handbooks for scribe schools. These “papyrus” (Develay, 1996) books enable the drawing of certain methodical conclusions: 1) There is a systematization of content. For example, the Rhind papyrus categorizes problems, according to their content, into problems of Arithmetics, problems of calculating areas and volumes, with an applicative nature; 2) There are no general formulas, but all the procedures are shown on concrete examples, like in a recipe book. 3) There occur the first elements of the “school” style, as elementary problems meant to show how certain algorithms should be applied are treated.

Here are several benchmarks from Hindu Mathematics: 1) There were handbooks, like the famous Suliva-Sutra (rules of the rope) which contain calculus rules in short stanzas meant to be memorized. 2) The drawings from geometrical “demonstrations” are often accompanied by the laconic urge: Look!

In Chinese Mathematics, fraction operations were performed on tables of calculations, according to detailed rules, in order to reduce fractions to an irreducible form. “Mathematics in nine books” contains rules and algorithms for calculations with numbers and fractions. Indians operated with squaring, operations with fractions and extractions of square and cube roots.

In Ancient Greece, pre-Hellenic Mathematics studied by philosophers became a deductive science whose results and characteristics still amaze men of sciences nowadays. Socrates established types of logical reasoning, sophisms and paradoxes highly appreciated by the scholars of Hellas. The first Greek mathematician was Thales of Millet, whose mathematical work is reduced only to Geometry and consists of the following sentences: 1. The angles at the basis of an isosceles triangle are equal; 2. Given a side and the adjacent angles of a triangle, this is completely determined (the application was determining a ship at sea seen from two points on the

shore); 3. Any diameter divides a circle into two equal semicircles; 4. Two triangles with respectively equal angles have proportional sides; 5. The angle inscribed in a semicircle is a right angle (it implies knowledge of the sum of angles in a triangle).

It is appreciated that all these results (formulas, procedures, solutions, results etc.) from Antiquity have become knowledge when demonstrations were introduced. Euclid's "Elements" served as a model for presentation for rational sciences, especially in Geometry until the 17th century. In "Elements", Euclid uses a way of treating theorems and problems which consists in: - stating the problem or the theorem; - mentioning the conditions which should be met; - building the figure; - demonstration; - deducing formulas, corollaries, lemmas.

The first Romanian text containing teaching methodology indications is found in the instructions of the *Regulamentul Organic* (1831), which probably belongs to Gheorghe Asachi. Even Eminescu recommended the use of intuitive material in teaching. Spiru Haret reorganizes educations at all levels by arguing that it should be closely connected to life and use intuition next to reasoning. We may speak of a scientific education of Mathematics, but only starting with a little more than a century ago.

Although there are no explicit books on the teaching methodology, we may notice didactic concerns at most of the Romanian Mathematicians. The famous "Collection of problems of Geometry", written by G. Țițeica at the start of the century (1904) was and still is a guide for learning Geometry. Traian Lalescu (author of the first book in the world on integral equations) also had the time and interest to look into elementary Geometry, publishing the book "Triangle Geometry".

After 1948, *Gazeta Matematică* split into *A Series*, containing numerous methodological articles, and *B Series*, addressed to student and containing, since 1980, articles on methodical and methodological training in Mathematics. An unforgettable name in this context is that of Eugen Rusu, who studied the school phenomenon (articles addressing teachers of Mathematics, the psychology of mathematical activities, problematization and problems in school Mathematics etc.).

To understand the evolution of mathematical education and of the concerns for the teaching-learning-evaluation of Mathematics, as well as the historical progress of the Didactics of Mathematics, we shall proceed to an analysis of the handbooks and syllabi from various countries.

Quadling D.A. gives an interesting description of the evolution of mathematical education reform, from the manifestations of the “first wave” of the 60s to the “return to bases” characteristic of the 80s and the subsequent modern trend based on curricular training. Reformers prescribed a regime of mathematical rigor. Students were placed in an aseptic environment and were administered mathematical structures on a regular basis. For the moment, it was believed that students succeeded in assimilating and understanding, but symptoms of poor training and even reluctance were soon identified. The reason for rejection was that many abstract notions were imposed as such from the outside, isolating the student from the outer world. Quadling also mentioned the fact that “we shall not establish mathematical health by contemplating ideal systems, but by active involvement of creation”.

Analysing the syllabi for Mathematics from Great Britain, France, Belgium and other countries, the mentioned author notices a rebound of the ambitious projects of the last decades and concludes that the current orientation of curricular reform is a transition from abstract and deductive presentation of mathematical structures to a theoretical and practical education placed on a less formal level.

The final report from the 4th International Congress of Mathematical Education, (Cambridge, 1983), mentions the fact that strict limits should be established regarding the mathematical contents which should be taught to students of different ages, taking into consideration the formative resources of the discipline in the context of the research undertaken at the level of school psychology (Haze et al., 2000, pp. 364-366).

The authors from the British group “School Mathematics Project”, who initially claimed that a special focus should be placed on the earliest understanding of algebraic structures, insisting upon the properties of operations, subsequently admitted to having been wrong, following significant criticism of their exaggeration in this respect, as well as of the fact that their first syllabi and handbooks allowed too little space to practicing calculus skills (Siebert, 2001, pp. 123-139). In Romania, after a four-year time span during which the handbooks of Geometry proposed by professor K. Teleman in the 78s-82s were used, they were eventually replaced for the same reasons.

One of the current directions of evolution of educational systems is the transition from a type of education intended for gifted students to a curricular education that may be efficient for all students and more connected to objective realities.

The modern socio-economic changes which imposed the generalization of the first stage of secondary education, usually up to the age of 16, as well as the example of the syllabi that aimed at introducing “Modern Mathematics” demand certain reconsiderations, re-evaluations, and re-adjustments of the design and achievement of mathematical education. The validity of this statement is confirmed by the fact that, during the 5th International Congress of Mathematical Education (Adelaide, Austria 1984), a team of researchers focused on the theme of “Mathematics for all”. In this context, in many countries there may be found a discrepancy between the objectives of existing mathematical education and the needs of most students. The phenomenon may be accounted for by the perpetuation, with slight changes, of the traditional syllabi of Mathematics, designed at the end of the 20th century, in the context of a mainly elitist education.

Conclusions

Generally, in early education or when students encounter difficulties, learning should be approached through cognitive structures of learning represented by actions and images. Bruner offers very important suggestions for elaborating a Didactics of science (Astolfi and Develay, 1990). There are features of instruction which express the construction requirements for a Didactics of science in the spirit of a curricular paradigm:

1. “A theory of instruction should indicate the experiments which represent the most effective way of implementing at a student the inclinations for general learning, or for a specific way of learning” (Example: see the general and specific objectives of Mathematics in relation to the school curriculum objectives).
2. “A theory of instruction should indicate the way in which a knowledge body should be structured in order to be understood by the learners” (see the school curriculum content that must have an optimal structure “which depends on its capacity to simplify information).
3. “A theory of training (of didactics) should determine the most efficient order to present the learning materials”. Thus, Science Didactics views methods as ways of punctual learning, but especially as strategies which offer more ample ways to articulate the knowledge on the long and middle term.
4. “A training theory should specify the nature and the process of the learning process”. It is about the integrated assessment from the learning process structure, which ensures the passage from the “extrinsic rewards (praise), to intrinsic rewards (inherent resolution independent of complex problems).

J.S. Bruner offers very useful suggestions for the development of discipline Didactics in the spirit of the curricular paradigm and of a constructivist model that complies with Science Didactics. We shall proceed to a historical incursion based on the study of several methodologies and didactics from the period 1990 – 2014, respectively from the period after the 90s. The research relies on the application of the historical research method through a synchronic approach (static in relation to the respective moment) and a diachronic approach (regarding the evolution of the respective works). Presentation of the structure and the author's motivation accompanied by our comments will be performed both synchronically (traditional didactics) as well as diachronically (in present terms, for the model of applied Didactics and the Didactics of science). We should also mention the famous people such as G. Polya, D. Barbilian, E. Rusu, R. Miron, D. Brânzei, S. Marcus, A. Hollinger, O. Popescu, V. Radu, I. Rus, D. Varna, A. Catană, M. Săcuiu, O. Stănășilă, Gh. Achiței, H. Banea, F. Cârjan, C. Lupu, D. Săvulescu, and others who, by their experience and educational activity, drew generalizing conclusions.

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