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HEURISTIC STRATEGIES FOR SOLVING MATHEMATICAL PROBLEMS

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Abstract

Our study aims at elaborating a methodological model that may fully exploit heuristic didactic strategies in the heuristic solving of problems. The organization of the study will focus on improving the use and efficiency of heuristic mathematical techniques by relating to heuristic problem solving. This article demonstrates the relevance of using heuristic problem-solving strategies in lessons of Mathematics, as a fundamental requirement with multiple valences in building thought operations, which leads to enhancing school performance. The research was conducted during the 2014-2015 school year, involving two groups, each of them comprising 24 students: experimental group – the 4th A grade from "Gura Văii" Middle School, a structure of "Ion Borcea" Middle School, Racova, Bacău, and a control group – the 4th B grade from "Gura Văii" Middle School, a structure of "Ion Borcea" Middle School, Racova, Bacău. Various methods were applied during the research: conversation, experiment, analysis of activity products, the method of the tests, statistical processing of the data.

Key words*:* Experiment, Mathematical heuristic techniques, problem situation

Introduction

The term *heuristic* comes from Greek: heuriskein – to find out, to discover. Heuristic teaching strategies represent mental exploitation strategies supporting the discovery of information, stimulate thought operations, the students' judgement and reasoning, leading to active, conscious learning. Traditional education, focused on the teacher and the learning content, has been replaced by modern, student-centred education. To achieve this desideratum, the teacher has to resort to heuristic teaching-learning strategies.

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The heuristic strategy implies a wide range of methods. This strategy represents the result of the interconditioning between the two components:

- *the teaching strategy* (elaborated by the teacher): the teacher's ability to select and combine, in a certain order, methods, procedures and training instruments, groupings of students, select and organize the scientific content according to the proposed objectives, opt for a certain learning situation that would be experienced by the students;
- *the learning strategy* (elaborated by the student), that may be: - participation strategies; - encoding strategies; - acquisition and reconstruction strategies; - strategies for elaborating hypotheses; - strategies connected to problem solving.

The teaching strategy creates circumstances for building the students' learning strategies and the learning methods determine the optimization of the teaching strategies. Schematically, our aim is to build a learning situation where the student learns (through guidance), builds (through semi-guidance) or elaborates (independently) strategies for learning the new content, solving strategies or even strategies for the self-guidance and control of one's own way of thinking. An essential element in elaborating the teaching strategy is selecting heuristic methods and procedures.

By heuristic method we mean a specific way for solving a general problem. It may include several procedures, these constituting details of the method, with a more limited sphere of applicability. The heuristic procedures may be defined as thought mechanisms that suggest and stimulate the generation of efficient conjunctures while solving the problem, or enable the shortening of the problem solving path.

The most frequently used heuristic methods include: the method of the analogy; generalization and particularization; analysis through synthesis; selecting, searching for a related problem; solving an auxiliary problem; rereading definitions; exploiting properties; reformulating the problem; demonstrative reasoning (deductive, inductive, analogical).

Theoretical aspects

Types of heuristic teaching strategies in problem solving

The teaching strategies highlight the teacher's ability to select and combine, in a certain order, training methods and procedures, groupings for students, select and organize the scientific content according to the proposed objectives, opt for a certain learning situation that will be experienced by the students.

The teaching strategy implies a certain way of approaching learning and teaching that may be: analytical or synthetic, intuitive or deductive, creative or algorithmic, theoretical or practical-applicative, frontal or individual, classic or modern, interdisciplinary or monodisciplinary.

According to the selected strategy, the teacher searches for and associates those operations (analysis, comparison, association, analogy, interpretation, generalization, abstraction etc.) in order to reach the desired acquisitions (knowledge, skills, abilities, behaviours, attitudes). In this respect, the students' physical and mental activity is decomposed into a series of sequences, with a view to organizing each moment of the lesson.

At the level of teaching, the strategy is part of the methodology, the teacher's art of leading, solving training situations. The teacher uses the elements of the teaching-learning-evaluation process as a system, in order to achieve objectives in a certain manner, a procedural option, a combinatory style, a coordination style, a model for typical and optimal solving. Therefore, it is an act of institutional management.

Characterized, essentially, as a way for combining and approaching teaching-learning-evaluation, of organizing the process in order to achieve objectives, the teaching strategy provides criteria for building training actions and situations by:

- selecting the orientation towards a certain type, form, way of teaching and learning and of conducting them;

- selecting the best set of methods, means, forms of organization that circulate the learning contents;

- indicating the conditions, minimal resources needed to reach a certain objective or group;

- conceiving, designing teaching-learning-evaluation sequentially or in compliance with a certain concatenation and order of these; finding the proper solution for defining, selecting, correlating the situations resulted from relating to previously established objectives;

- achieving various combinations of these elements of the process of training, both at the global level (macro-design) and at the level of a concrete teaching, learning (micro-design) situation, in relation to a certain operational objective;

- indicating a certain way for introducing the student into the created situation, guiding him in solving the task, until its completion and evaluation;

- relating this combination to other determined conditions – the students' initial level of training, allotted time, moment of beginning, place among the other situations, material circumstances;

- formulating a version, a solution resembling a decision, after having processed the information accumulated in relation to the elements of the situation, such as type, organization and conducting it;

- the possibility to particularize its elements into actions, delimited operations (procedures) that may enhance the degree of precision, control, prevention of deviations and streamlining;

- the teacher's possibility to guide the situation's evolution, to seize disturbing factors and intervene, to find solutions for adopting or selecting another method ad-hoc;

- to engage students according to their particularities, to assert creativity, teaching style and how the teacher leads the action;

- indicating the proper way for putting the student into contact with the objectives, contents, concrete tasks, achievement conditions, evaluation criteria, the type of learning and exploiting previous experience;

- formulating even research hypotheses for optimizing training by introducing, experimenting new methodological, organizational combinations;

- delimiting the degree and form for extending the guidance of students in training, solving, generalizing results, involving them in specific learning activities;

- supporting the teacher in finding answers to the questions he himself raises while designing teaching, defining and combining the required training-evaluation situations;

- unifying criteria, adjusting them to the establishment of the strategy for solving the situation: the teacher's design, objectives, informational content, the students' type of experience, the rules that must be complied with, the teaching-material resources, the allotted time.

Types of heuristic strategies in solving problems

a) *according to the learning activity in the training process*:

- **algorithmic**: through imitation of given models; through repetition, practice, memorization; through reception, reproduction; through concrete-intuitive knowledge; through algorithmization, step by step;

- **heuristic**: through unmediated observation; by solving open problems ; through experimentation ; through debates, heuristic dialogues; through group research; through simulation, modelling, applications; through creativity techniques etc.

- mixed: by combining all the other types.

b) according to the way of guiding learning: step-by-step guidance; semi-guidance; partial non-intervention.

c) according to the type of reasoning applied: inductive teaching-learning; deductive teaching-learning; transductive teaching-learning; learning by analogy; combining reasonings.

Any strategy is simultaneously a technique and educational art, the selection and use of any type of strategy decisively depends on the teacher's training and personality, since during a teaching activity the teacher may use a combination of strategies, corresponding situations in order to enhance the efficiency of his actions and the quality of results.

Methodology

Research objectives

The research objectives were:

1. Knowledge of the heuristic teaching methods in order to be able to heuristically solve problems by studying the reference bibliography and the experience achieved during lessons of Mathematics;
2. Understanding the school syllabus for grades I-IV;
3. Elaborating (initiating) a personal methodological process to fully exploit heuristic teaching strategies;
4. Organizing and conducting the experiment (in order to achieve the proposed objectives);
5. Analysing, processing and presenting the obtained results (in order to demonstrate, in an efficient way, the heuristic methods used in problem solving);
6. Formulating conclusions (in order to understand the efficiency of the experiment).

The research hypothesis

The organization of our experiment relied on the following hypothesis: *If during the act of teaching-learning there are efficiently used heuristic mathematical problem-solving strategies, with multiple formative valences in building thought operations, then these will generate an increase in school performance and the students' results will be much improved.*

Procedure

The experiment was conducted during the 2014-2015 school year, at the 4th grade from "Gura Văii" Middle School, a structure of "Ion Borcea" Middle School, Racova, Bacău. This class comprises a number of 24 students. The research was conducted during the 2014-2015 school year, involving two groups, each of them comprising 24 students: experimental group – the 4th A grade from "Gura Văii" Middle School, a structure of "Ion Borcea" Middle School, Racova, Bacău, and a control group – the 4th B grade from "Gura Văii" Middle School, a structure of "Ion Borcea" Middle School, Racova, Bacău.

The stages of the experiment

The stage of initial evaluation aimed at observing the students' level of training by applying initial testing which consisted of observation protocols and a knowledge test (comprising different exercises and problems).

The stage of formative-ameliorative evaluation, during which there was introduced the progress factor and there were varied the manifestation circumstances by using active heuristic teaching methods, besides those used in the heuristic problem solving process.

The stage of final evaluation consisted in a comparison of the results obtained in the initial test, in order to highlight the students' progress/ regress at lessons of Mathematics, especially in problem solving.

The research variables are:

- the independent (introduced) variable, namely the use of active teaching methods;
- the dependent variable that leads to enhancing the efficiency of heuristic methods of solving problems and the students' school progress.

Research methods

The research relied on the following knowledge methods and techniques**:**

1. The method of observation that is frequently used in school. Both spontaneous (passive) and scientific (generated) observation support the accumulation of a rich factual material, being able to provide data on the students' behaviour during lessons, breaks, extra-curricular and family activities.

2. The method of conversation was used to gather information from students, parents, the family's general practitioner or other acquaintances of the students. Thus, we could access data on the students' interests and aspirations, temperamental particularities, character features, general intelligence, family climate, material circumstances, daily regime, health, hobbies, likes/ dislikes in relation to certain activities, possibilities for doing homework.

3. The psychological analysis of the activity's results/ products provides information on several aspects related to the products of the activity. The data collected through this method was analysed by detaching appreciations and estimations related to the students' individuality, behaviour, inclinations and interests, the way in which they do their homework, their concern for correctness.

4. The method of tests represents a method that supports the diagnosis of the subject's development level – in this case, students – and formulating, on this basis, a prognosis regarding their evolution. *Docimological tests* provide quantitative information on the investigated phenomenon, when applied regularly during the instructive-educational process from the classes of Mathematics, as well as from other disciplines, have supported the determination of the level of knowledge, skills and the level of development of intellectual skills. They were conceived in relation with the established operational objectives, comprising sets of items meant to help us record and evaluate school performances.

5. The statistical-mathematical methods were used to analyse the obtained results that were inserted in analytical and synthetic tables, then systematized in centralized tables, graphs, histograms, circular diagrams, supporting the interpretation of data.

Research results

Initial evaluation

During the observational stage, we applied an initial evaluation test. The test was elaborated by taking into account the objectives that had to be achieved by the end of the 4^{th} grade, in order to establish the student's level of training.

Analysing the data from the tables, we may argue that:

- the results obtained by the students from the experimental class constitute information on the knowledge of the respective student, as well as the student's knowledge gaps;

- the total score at the level of the class represents the sum of the points obtained for each item plus one point from the office.

Following the recording of these data, our conclusions regarding the students' initial training level are the following: the students had difficulties in solving problems; the average of the experimental class is 7.3, this representing the starting point in conducting our research.

The initial test was meant to establish the students' level of training. The test helped us notice the fact that the most difficult item was I_4 , whereas the best results were obtained at items I_1 , I_2 , I_3 . The data per student demonstrated relevant differences between the students who had solved 2-3 tasks and those who had solved all the tasks. We found that the level of the class is lower-intermediate.

Applying the initial test enabled us to identify the students' learning difficulties in the initial phase and, in relation to their extent, a more prolonged focus on the respective content until all the students have achieved a corresponding training level.

Analysing the graphs that represent the results obtained by the students from the *experimental class*, we found that from the 24 evaluated children, 11 obtained the mark VW (very well) representing 43%, 8 children obtained the mark W (well), representing 36%, and 5 children obtained the mark S (sufficient), representing 21% of the participants.

Analysing the graphs that show the results obtained by the students of the class, we found that in the initial evaluation, the results of the *control group* were the following: from the 24 evaluated children, 11 obtained the mark VW (very well), representing 43% of them, 7 children obtained the mark of W (well), representing 30%, and 4 children obtained the mark S (sufficient), representing 16%, whereas 2 children obtained the mark I (insufficient), representing 11 % of the participants.

Analysing the results obtained by the students with poorer results, we found that these are challenged by difficulties in solving the following tasks: - they do not perform calculi correctly; - they do not solve problems completely; - they do not compose problems following the given model; - they do not find the question that they need to raise in order to solve the problem.

Following the results obtained by the experimental class, we have noticed the fact that most students come across difficulties when solving problems.

Formative evaluation

The formative evaluation tests applied during lessons of Mathematics enabled the immediate knowledge of the students' learning difficulties. In order to eliminate the errors, we resorted to differentiating the activities. Following the analysis of the tests, there were presented the unachieved operational objectives, so that these may be aimed at during the proposed recovery activities.

Analysing the data, we may argue that although the students from the experimental class did not record major leaps in terms of their marks, almost all of them achieved better scores compared to the previous tests, therefore the learning experience was a success. We have also noticed the fact that the most frequent errors were those related to calculus, which indicates that the methods used in the heuristic solving of problems are known and acquired by the students of the class.

The formative evaluation tests applied during lessons of Mathematics enabled the immediate identification of errors and the students' learning difficulties. Looking at the tables with the data from the ameliorative formative tests and at the graphs with the scores and marks obtained in the initial tests, we may notice the fact that the school performance was improved as follows: - the average at the initial test for the experimental group was 7.3 and at formative test no. 1 the average was 8; at formative test no. 2, there was a slight increase compared to the first test, the average being 8.2.

This increase is due to surpassing the more serious difficulties related to the contents of learning. The scores obtained were significantly higher than for the previous test. The results obtained highlight the relevance of formative evaluation tests applied during the learning activities and confirm the usefulness of the heuristic methods used. The fact that the results of the students from the experimental class were improved, with even the less industrious students achieving a promotion level, determined us to interfere, when it was necessary, with worksheets for repeating certain tasks, in order to achieve a more thorough acquisition of knowledge.

The progress obtained by the students compared to the initial test cannot be interpreted only as enhancement of percentages related to achieving objectives, but also in relation to the use of heuristic working methods, which led to activating the desire for performance or for increasing performance and, implicitly, a more active, conscious participation of students.

Summative (final) evaluation

On May the 1st 2015, there was applied the final evaluation of students through an evaluation test. In order to centralize and interpret the data, we have resorted to analytical and synthetic tables, frequency polygons, histograms and diagrams. The final evaluation test was designed in a similar manner to the initial one, so that the results obtained may be compared, the knowledge included in the syllabus being defined as operational objectives encoded as items.

The analysis of the analytical and synthetic tables of the histogram, the frequency polygon and the circular diagram revealed the fact that in the final evaluation, for the *experimental group*, the results were the following: from the 24 evaluated children, 15 obtained the mark VW (very well), representing 64%, 8 children obtained the mark W (well), representing 29 %, and 1 child obtained the mark S (sufficient), representing 7 % of the participants.

The analysis of the analytical and synthetic table, of the histogram, frequency polygon and circular diagram, revealed that in the final evaluation, the results for the experimental group were the following: from the 24 evaluated children, 11 obtained the mark VW (very well), representing 44%, 9 children obtained the mark W (well), representing 37%, whereas 4 children obtained the mark S (sufficient), representing 19% of the participants.

The comparative analysis of the data obtained in the initial and final evaluation form

In order to highlight the progresses related to improving relations following the conducted experiment and the applied methodology, we have proceeded to performing a comparative analysis of the two series from the initial and final evaluation.

Comparative analysis for the experimental group

The comparison of the results obtained in the predictive and final test have revealed the fact that throughout the school year, as a result of the systematic application of active methods and differentiated learning during lessons, the progress of students was both qualitative and quantitative. This fact was easily seen in the ease and pleasure with which the students acquired a great amount of knowledge, with which they operated in solving problems and problem-situations (knowledge acquired especially through their personal effort), in the pleasure with which they worked throughout the entire school year (Table 1, Figure 1).

Table 1. *The comparative analysis of the results at the level of the experimental group*

MARKS	Initial evaluation	Final evaluation
VERY WELL	11	15
WELL	8	8
SUFFICIENT	5	1
INSUFFICIENT	0	0

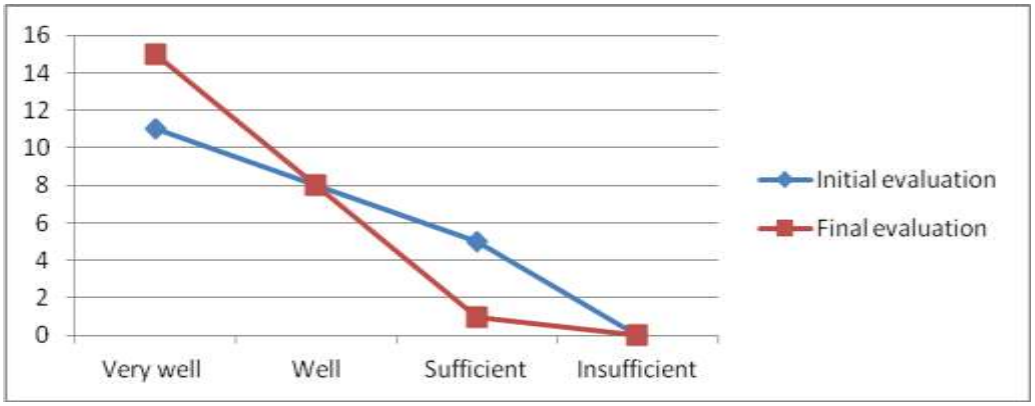

Figure 1. *The graphical representation of the results obtained by the for the students from the experimental group*

The comparative analysis of the table and frequency polygon revealed the progress recorded at the end of the experiment by the experimental group. The results obtained in the final evaluation show an obvious difference from the scores obtained in the initial evaluation. This reveals the fact that the formative stage was efficient, the results obtained demonstrating the improvement of the results.

Comparative analysis for the control group

The comparative analysis of the table and frequency polygon reveals, for the control group, the fact that the number of students who obtained the mark VW remained the same the number of those who obtained the mark W increased, the number of those with mark S did not increase but there increased the percentage for mark I. The results obtained in the final evaluation test did not increase significantly compared to the points obtained at the stage of initial evaluation (Table 2, Figure 2).

Table 2. *Comparative analysis of the results at the level of the control group*

MARKS	Initial evaluation	Final evaluation
WERY WELL	11	11
WELL	7	9
SUFFICIENT	4	4
INSUFFICIENT	2	0

Figure 2. *The graphical representation of the results obtained by the for the students from the control group*

Comparative analysis between the two groups in the final evaluation

The comparative analysis of the histogram and frequency polygon reveals the progress recorded at the end of the experiment by the experimental group. Calculating the average between the two tests (initial and final) and drawing a comparison between the two groups, there may be observed an increase in the school performance for the experimental group as compared to the control group (Table 3, Figure 3).

Table 3. *Comparative analysis of the results*

MARKS	EXPERIMENTAL GROUP	CONTROL GROUP
VERY WELL	15	11
WELL	8	9
SUFFICIENT	1	4
INSUFFICIENT	0	0

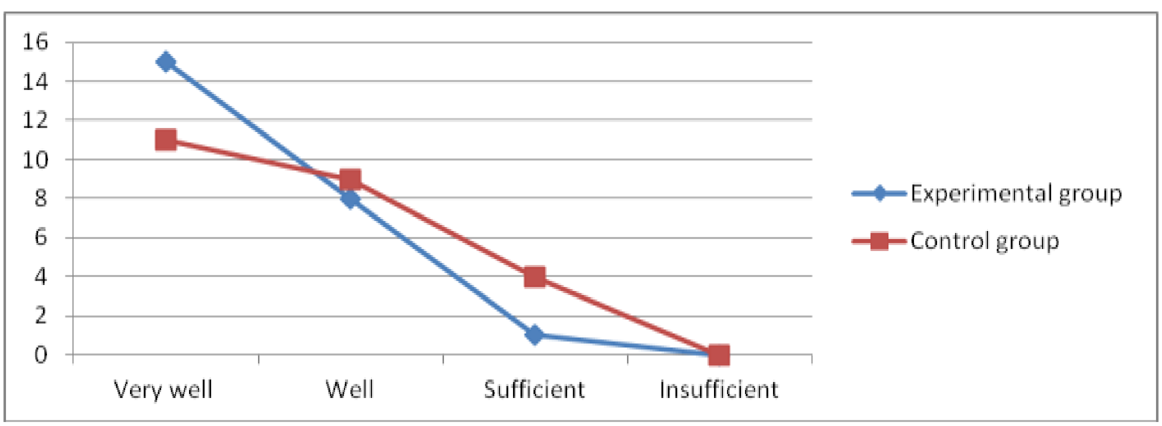

Figure 3. *The graphical representation of the results*

Conclusions

In general, it may be said that solving problems constitutes the most appropriate way for achieving the objectives of teaching-learning Mathematics. The activity of Mathematics requires effort, focus and activation of all the components of the human psychic, particularly thought and intelligence. The intellectual effort put into composing and solving problems is, essentially, a continuous exercise that results in building the students' imagination and creativity.

From the instructive-educational perspective, solving problems constitutes the application of acquired knowledge in relation to mathematical operations and their properties, deepening and consolidating knowledge. In terms of practice, solving problems represents the seizing and understanding of the relations between sizes that we come across on a daily basis, for the solving of which it is not enough to know only the calculus technique.

The main objective of each lesson should serve not just training, but also education, an action where the leading role belongs to the educator. This should avoid the formal nature of the lesson and ensure an atmosphere of constant communication, the students participating with their own ideas, questions that the educator should tactfully guide towards the proposed educational goal. At the same time, he should aim at the accessibility of learning by challenging the student, in a systematic, conscious, gradated way, with obstacles that the student may overcome under his guidance.

Composing and solving problems will challenge students throughout the entire school period as well as their entire life, but by being discreetly led towards discovering the solution, they will be enthusiastic and encouraged to obtain more and more performances.

The results obtained by applying the tests have generated the following findings:

- the data obtained highlighted the higher results from the final test compared to the initial test, demonstrating the efficiency of the systematic mental training in finding several alternatives for solving a problem;

- the continuous, sustained solving of problems also helped the students with poorer results, removing their fear of failure and shyness;

- the systematic training of students in finding as many possible alternatives for solving a problem leads to building the students' creativity;

- involving the students in creative, active-participative activities gives the teacher the possibility to know individual particularities better, the style of each student, intelligence, will, temperament, behaviour, in a word, personality.

I believe that the proposed objective and hypothesis have been confirmed, our work constituting a possible guide for teachers in their activity of solving simple, composed or typical problems.

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